The free group in **R**: introducing the freegroup package

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Abstract

Here I present the `freegroup` package for working with the free group on a finite set of symbols. The package is vectorised; internally it uses an efficient matrix-based representation for free group objects but uses a configurable print method. A range of R-centric functionality is provided. It is available on CRAN at `https://CRAN.R-project.org/package=freegroup`. To cite the **freegroup** package, use Hankin (2022).

Keywords: Free group, Tietze form.

1. Introduction

The free group is an interesting and instructive mathematical object with a rich structure that illustrates many concepts of elementary group theory. The **freegroup** package provides some functionality for manipulating the free group on a finite list of symbols. Informally, the *free group* $(X, \circ)$ on a set $S = \{a, b, c, \ldots, z\}$ is the set X of *words* that are objects like $W = c^{-4}bb^2aa^{-1}ca$, with a group operation of string juxtaposition. Usually one works only with words that are in "reduced form", which has successive powers of the same symbol combined, so W would be equal to $c^{-4}b^3ca$; see how b appears to the third power and the a term in the middle has vanished. The group operation of juxtaposition is formally indicated by $\circ$, but this is often omitted in algebraic notation; thus, for example $a^2b^{-3}c^2 \circ c^{-2}ba = a^2b^{-3}c^2c^{-2}ba = a^2b^{-2}ba$.
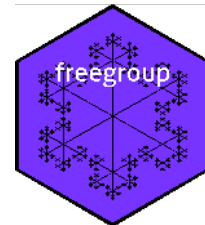
1.1. Formal definition

If X is a set, then a group F is called *the free group on X* if there is a set map $\Psi\colon X \longrightarrow F$, and for any group G and set map $\Phi\colon X \longrightarrow G$, there is a unique homomorphism $\alpha\colon F \longrightarrow G$ such that $\alpha \circ \Psi = \Phi$, that is, the diagram below commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \Psi\ } & F \\
& \searrow{\scriptstyle\Phi} & \downarrow{\scriptstyle\alpha} \\
& & G
\end{array}
$$

It can be shown that F is unique up to group isomorphism; every group is a quotient of a free group.

1.2. Existing work

Computational support for working with the free group is provided as part of a number of algebra systems including GAP, Sage (The Sage Developers 2019), and sympy (Meurer *et al.* 2017) although in those systems the emphasis is on finitely presented groups, not in scope for the **freegroup** package. There are also a number of closed-source proprietary systems which are of no value here.

2. The package in use

In the **freegroup** package, a word is represented by a two-row integer matrix; the top row is the integer representation of the symbol and the second row is the corresponding power. For example, to represent $a^2b^{-3}ac^2a^{-2}$ we would identify a as 1, b as 2, etc and write

```
> (M <- rbind(c(1,2,3,3,1),c(2,-3,2,3,-2)))


     [,1] [,2] [,3] [,4] [,5]
[1,]    1    2    3    3    1
[2,]    2   -3    2    3   -2
```

(see how negative entries in the second row correspond to negative powers). Then to convert to a more useful form we would have

```
> library("freegroup")
> (x <- free(M))


[1] a^2.b^-3.c^5.a^-2
```

The representation for R object `x` is still a two-row matrix, but the print method is active and uses a more visually appealing scheme. The default alphabet used is `letters`. We can coerce strings to free objects:

```
> (y <- as.free("aabbbcccc"))


[1] a^2.b^3.c^4
```

The free group operation is simply juxtaposition, represented here by the plus symbol:

```
> x+y


[1] a^2.b^-3.c^5.b^3.c^4
```

(see how the a "cancels out" in the juxtaposition). One motivation for the use of "+" rather than "*" is that Python uses "+" for appending strings:

```
>>> "a" + "abc"
'aabc'
>>>
```

However, note that the "**+**" symbol is usually reserved for commutative and associative operations; string juxtaposition is associative. Multiplication by integers—denoted in **freegroup** idiom by "*∗*"—is also defined. Suppose we want to concatenate 5 copies of x:

```
> x*5
```

```
[1] a^2.b^-3.c^5.b^-3.c^5.b^-3.c^5.b^-3.c^5.b^-3.c^5.a^-2
```

The package is vectorized:

```
> x*(0:3)
```

```
[1] 0                              a^2.b^-3.c^5.a^-2
[3] a^2.b^-3.c^5.b^-3.c^5.a^-2     a^2.b^-3.c^5.b^-3.c^5.b^-3.c^5.a^-2
```

There are a few methods for creating free objects, for example:

```
> abc(1:9)
```

```
[1] a               a.b             a.b.c           a.b.c.d
[5] a.b.c.d.e       a.b.c.d.e.f     a.b.c.d.e.f.g   a.b.c.d.e.f.g.h
[9] a.b.c.d.e.f.g.h.i
```

And we can also generate random free objects:

```
> rfree(10,4)
```

```
 [1] b^-2.d^4        d^-4.a.d^3      c.b.a^-6         b^5.d^3.b^-2
 [5] c^4.d^-6        b^3.d.b^3       d^-4.b^3.c^-4.a  b^-1.a^4.d^2.a^3
 [9] d^-1.a^2        c.b^-3.d
```

Inverses are calculated using unary or binary minus:

```
> (u <- rfree(10,4))
```

```
 [1] d.b^-2            d                 a^3.b^4         d^3.b^4.a^3
 [5] a^-2.d^-1.b.d^-1  b^4.c^2.a^-3      b^2.d^3.a^4.d^-1 d^-2.c^-6.a^-2
 [9] a^-1              a^-5.d^-1.b^2
```

```
> -u
```

```
 [1] b^2.d^-1          d^-1            b^-4.a^-3        a^-3.b^-4.d^-3
 [5] d.b^-1.d.a^2      a^3.c^-2.b^-4   d.a^-4.d^-3.b^-2 a^2.c^6.d^2
 [9] a                 b^-2.d.a^5
```

```
> u-u
```

```
 [1] 0 0 0 0 0 0 0 0 0 0
```

We can take the "sum" of a vector of free objects simply by juxtaposing the elements:

```
> sum(u)
```

```
[1] d.b^-2.d.a^3.b^4.d^3.b^4.a.d^-1.b.d^-1.b^4.c^2.a^-3.b^2.d^3.a^4.d^-3.c^-6.a^-8.d^-1.b^
```

Powers are defined as per group conjugation: x^y == y^{-1}xy (or, written in additive notation, −y+x+y):

```
> u
```

```
 [1] d.b^-2            d               a^3.b^4          d^3.b^4.a^3
 [5] a^-2.d^-1.b.d^-1  b^4.c^2.a^-3     b^2.d^3.a^4.d^-1 d^-2.c^-6.a^-2
 [9] a^-1              a^-5.d^-1.b^2
```

```
> z <- alpha(26)
> u^z
```

```
 [1] z^-1.d.b^-2.z          z^-1.d.z              z^-1.a^3.b^4.z
 [4] z^-1.d^3.b^4.a^3.z     z^-1.a^-2.d^-1.b.d^-1.z z^-1.b^4.c^2.a^-3.z
 [7] z^-1.b^2.d^3.a^4.d^-1.z z^-1.d^-2.c^-6.a^-2.z   z^-1.a^-1.z
[10] z^-1.a^-5.d^-1.b^2.z
```

Thus:

```
> sum(u^z) == sum(u^z)
```

```
[1] TRUE
```

There is also a commutator bracket, defined as $[x,y] = x^{-1}y^{-1}xy$ or in package idiom .[x,y]=-x-y+x+y:

```
> .[u,z]
```

```
 [1] b^2.d^-1.z^-1.d.b^-2.z
 [2] d^-1.z^-1.d.z
 [3] b^-4.a^-3.z^-1.a^3.b^4.z
 [4] a^-3.b^-4.d^-3.z^-1.d^3.b^4.a^3.z
```

```
 [5] d.b^-1.d.a^2.z^-1.a^-2.d^-1.b.d^-1.z
 [6] a^3.c^-2.b^-4.z^-1.b^4.c^2.a^-3.z
 [7] d.a^-4.d^-3.b^-2.z^-1.b^2.d^3.a^4.d^-1.z
 [8] a^2.c^6.d^2.z^-1.d^-2.c^-6.a^-2.z
 [9] a.z^-1.a^-1.z
[10] b^-2.d.a^5.z^-1.a^-5.d^-1.b^2.z
```

If we have more than 26 symbols the print method runs out of letters:

```
> alpha(1:30)
```

```
 [1] a  b  c  d  e  f  g  h  i  j  k  l  m  n  o  p  q  r  s  t  u  v  w  x  y
[26] z  NA NA NA NA
```

If this is a problem (it might not be: the print method might not be important) it is possible to override the default symbol set:

```
> options(freegroup_symbols = state.abb)
> alpha(1:30)
```

```
 [1] AL AK AZ AR CA CO CT DE FL GA HI ID IL IN IA KS KY LA ME MD MA MI MN MS MO
[26] MT NE NV NH NJ
```

3. Conclusions and further work

The **freegroup** package furnishes a consistent and documented suite of reasonably efficient R-centric functionality. Further work might include the finitely presented groups but it is not clear whether this would be consistent with the precepts of R.

References

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