Threshold cointegration: overview and implementation in R

Matthieu Stigler

Matthieu.Stigler at gmail.com

January 10, 2010 (Revision 6: Nov 2020)

For an updated version, with emphasis on the Generalized Impulse Function (GIRF), see:
Stigler (2020) *tsDyn: Threshold cointegration: overview and implementation in R*
Handbook of Statistics, 42:229-264, Elsevier, www

Abstract

Purpose of this paper is twofold. It is first to offer a rough overview on the field of threshold cointegration, from the seminal paper of Balke and Fomby (1997) to the recent developments. Simultaneously, it is to describe the implementation of the main functionalities for the modeling in the open-source package `tsDyn`. It provides hence a unique way to get an introduction on the threshold cointegration field allowing in the same time to conduct its own analysis.

Introduced by Engle and Granger (1987), the concept of cointegration became a indispensable step in the analysis of non stationary time series. The underlying idea is that even if two variables (or more) are non-stationary, there can exist a combination of them which is stationary. This definition leads to interesting interpretations as the variables can then be interpreted to have a stable relationship (a long-run equilibrium), can be represented in an vector error-correction model, and share a common stochastic trend.

However, implicit in the definition is the idea that every small deviation from the long-run equilibrium will lead instantaneously to error correction mechanisms. Threshold cointegration extends the linear cointegration case by allowing the adjustment to occur only after the deviation exceed some critical threshold, thus taking into account possibly transaction costs or stickiness of the prices. Furthermore, it allows to capture asymmetries in the adjustment, where positive or negative deviations won't be corrected in the same manner.

Contents

1 Introduction: linear cointegration

On stationarity When dealing with time series, a main concern for statistical analysis is stationarity, the usual inference being based on that assumption. In its weak version, stationarity is defined as the finiteness and time-invariance of the expectation, variance and auto-covariance of a series. However, there are considerable theoretical (Samuelson, 1965) and empirical (Nelson and Plosser, 1982) arguments in favor of non-stationarity of economic series, especially for the difference-stationary type. A difference-stationary (or *integrated*) series is defined as a series that is non-stationary, but whose difference is stationary, as the random walk is. Great care should be taken when analyzing such series as they follow a different asymptotic behavior and particularly, regression among integrated series leads to a so-called spurious regression, i.e. inflation of regression indicators (t-tests, R^2) which lead to the false conclusion of statistical dependence between the series (Granger and Newbold 1974, Philipps 1986). An obvious remedy is to use differenced series, for which usual asymptotics apply. This approach has become the standard in the VAR framework popularized by Sims (1980).

Cointegration Granger introduced in 1982 the concept of cointegration, which means integrated series for which a linear combination exists that is stationary. This can be interpreted economically as the presence of a long-run equilibrium, the relationship between the variables being stable. The concept gained a significant interest with the so-called Granger representation theorem, which states that cointegrated variables have a vector error correction model (VECM) representation, that can be seen as a VAR model including a variable representing the deviations from the long-run equilibrium. Equation1 shows a VECM for two variables including a constant, the error-correction term and a lag.

$$\begin{bmatrix}\Delta X_t \\ \Delta Y_t\end{bmatrix} = \begin{bmatrix}c_1 \\ c_2\end{bmatrix} + \begin{bmatrix}a_1 \\ a_2\end{bmatrix} ECT_{-1} + \begin{bmatrix}b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix} \begin{bmatrix}\Delta X_{t-1} \\ \Delta X_{t-1}\end{bmatrix} + \begin{bmatrix}\varepsilon_t^X \\ \varepsilon_t^Y\end{bmatrix} \qquad ECT_{-1} = (1, -\beta) \begin{bmatrix}X_{t-1} \\ Y_{t-1}\end{bmatrix} \tag{1}$$

This VECM representation is particularly interesting as it allows to estimate how the variables adjust deviations towards the long-run equilibrium, to test for Granger-causality as well as to determine the impacts of shocks to the variables using impulse response functions. In a system with more than $k > 2$ variables, there may exist $k - 1$ cointegrating relationships, hence the vector A of adjustment speed parameters (also called loading matrix) and the vector B of cointegrating values become matrices[1]. The matrix of their product, corresponding to the parameters of the lagged vector, is singular with rank equal to the number of cointegrating relationships.

Two methods captured the main attention and are of popular use now. The first one was advocated by Engle and Granger (hereafter E-G), who propose a two-steps approach, estimating the cointegrating values in the long-run representation and then plugging those estimates one the VECM representation[2]. The related testing procedure taking absence of cointegration as a null hypothesis consists in determining whether the residuals from the first step are stationary or not. Rejection of the stationarity is then interpreted as the

[1] of dimension $r \times k$, with r the number of cointegrating values and k the number of variables

[2] This two-step approach has been justified afterwards by the fact that the estimator in the first step is super-consistent, i.e. converging to its true value at rate n instead of usual rate $\sqrt{n}$ (Stock, 1987)

rejection of the null hypothesis of cointegration. When the cointegrating vector is known, usual unit root tests can be applied, whereas in case it is unknown, different critical values need to be used. Philipps and Ouliaris (1990) developed a test that is invariant to the normalization available in the software R in package *urca* (Pfaff 2008a).

Estimation and testing A major drawback of the E-G approach is that is allows to estimate and test only for one cointegrating relationship. When the cointegrating vectors are known, estimation of multiple cointegration relationship is trivial as the estimation is simply OLS regression for the VECM and testing can be done using classical Wald tests (Horvath and Watson 1995). When these vectors are unknown, the reduced-rank approach adopted by Johansen (1998, 1995)Johansen (1988, 1995) is able to estimate the potential cointegrating vectors and to test for their significance, allowing to determine the number of cointegration relationships. This is available in R with the *ca.jo()* function from package *vars* (Pfaff 2008b).

2 The extension to threshold cointegration

Balke and Fomby (1997) note that in the concept of cointegration there is the implicit assumption that the adjustment of the deviations towards the long-run equilibrium is made instantaneously at each period. There are nevertheless serious arguments in economic theory to invalidate this assumption of linearity. Among them, the presence of transaction costs is maybe the most noteworthy, as it implies that adjustment will occur only once deviations are higher than the transactions costs, and hence adjustment should not happen instantaneously and at each time. Financial theory predicts that even in highly liquid markets a so-called band of no arbitrage may exist where deviations are too small for the arbitrage to be profitable. In the domain of macroeconomics, policies are often organized around targets, where intervention is activated only once the deviations from the target are significant, the most famous example being the monetary policy management during the Bretton Woods agreement where central banks pegged their exchange rate and allowed a +/- 1 % band.

A second range of arguments that were raised in favour of nonlinear adjustment concerns the assumption of symmetry. In the linear cointegration context, increases or decreases of the deviations are deemed to be corrected in the same way. Again, several theoretical arguments may contest this assumption, such as the presence of menu costs (Levy, Bergen, Dutta, and Venable, 1997, Dutta, Bergen, Levy, and Venable, 1999), market power (Damania and Yang, 1998, Ward, 1982) or simply small country vs rest of the world effects.

Balke and Fomby (1997) introduced the concept of threshold cointegration, which allows to take into consideration the two main criticisms (though BF were concerned only with the first one) raised against linear cointegration. In their framework, the adjustment does not need to occur instantaneously but only once the deviations exceed some critical threshold, allowing thus the presence of an inaction or no-arbitrage band. They base their adjustment process on the self-exciting threshold autoregressive model (SETAR[3]) introduced by Chan (1983) and discussed extensively in Tong (1990). In the SETAR model, the autoregressive coefficients take different values depending on whether the previous value is above or under a certain threshold value, thus exhibiting regime switching dynamics. Hence, the linear adjustment process:

[3]But they call this model TAR, which is a more general form presented later.

$$\varepsilon_t = \rho\varepsilon_{t-1} + u_t \tag{2}$$

is extended as:

$$\varepsilon_t = \begin{cases} \rho_L\varepsilon_{t-1} + u_t & \text{if} \quad \varepsilon_{t-1} \leq \theta_L \\ \rho_M\varepsilon_{t-1} + u_t & \text{if} \quad \theta_L \leq \varepsilon_{t-1} \leq \theta_H \\ \rho_H\varepsilon_{t-1} + u_t & \text{if} \quad \theta_H \leq \varepsilon_{t-1} \end{cases} \tag{3}$$

This is actually a piecewise linear model where three different AR(1) processes are estimated depending on the state of the variable at time $t-1$. Autoregressive parameters are denoted with subscript L, M and H standing for Low, Middle and High regime, and they differ whether the variable was below the lower threshold θ_L, between the lower θ_L and upper θ_H threshold, or above the higher θ_H. This leads to some further remarks:

- The threshold effect is present when $\rho_H \neq \rho_M$and $\rho_L \neq \rho_M$ and as long as $0 < P(\epsilon < \theta_b) < 1$ where $b = L$ or H
- The SETAR model nest the AR when $\rho_H = \rho_M = \rho_L$.
- The model can easily by extended by adding lags in each regime, as well as intercepts.
- Several restricted models have been proposed, the main restriction being that the outer regimes are symmetric ($\theta_L = \theta_M$ along with $\rho_H = \rho_L$).

While the work of Balke and Fomby (1997) focused on the long-run relationship representation, extension to a threshold VECM (TVECM) has been made by several authors, the threshold effect being applied the anticipation by the agents of interventionary policy only to the error-correction term (Granger and Lee 1989, Seo 2006) or to the lags and the intercept as well (Hansen and Seo 2002, Lo and Zivot 2001).

$$\begin{bmatrix} \Delta X_t \\ \Delta Y_t \end{bmatrix} = \begin{bmatrix} c^X \\ c^Y \end{bmatrix} + \begin{cases} \begin{bmatrix} a_{XL} \\ a_{YL} \end{bmatrix} ECT_{L,t-1} \\ \begin{bmatrix} a_{XM} \\ a_{YM} \end{bmatrix} ECT_{M,t-1} \\ \begin{bmatrix} a_{XH} \\ a_{YH} \end{bmatrix} ECT_{H,t-1} \end{cases} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \Delta X_{t-1} \\ \Delta Y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^X \\ \varepsilon_t^Y \end{bmatrix} \tag{4}$$

In this model, the error-correction term is split into three regimes, lower (L), middle (M) and high (H) depending on whether it is below, between or above the two thresholds θ_L and θ_H, as in 3.

Note that speaking of threshold cointegration in the case of the TVECM was rather a conjecture as no formal representation theorem had been demonstrated in the threshold case. Nevertheless, the intuition of these authors was justified as Krishnakumar and Netto (2009) derived recently such a theorem, under the assumption that the loading matrix (the matrix of coefficients a in 4) is diagonal. An interesting conclusion of their theorem is that the threshold effect is present on the lags also only if the residuals from the LR

follow a multivariate SETAR process with one lag in each regime. When the residuals follow a multivariate SETAR process with p >1 lags, the lags in the TVECM do also have regime-specific components.

Several other model specifications have been used in the literature. Gonzalo and Pittarakis(2006a) and Krishnakumar and Netto (2009) use as transition variable not the deviations itself, but an external variable (that is, a TAR model, as we will see later). This has important implications as the stationarity conditions are different, estimation is much easier, and testing is more restrictive[4]. In that sense, it should be clearly differentiated from threshold cointegration as introduced by BF and I will denote it by cointegration with threshold effects. In my opinion, this approach is less attractive as it lets unanswered the question why an influencing variable is not included in the VECM.

All previous studies were based on the definition of threshold cointegration as a case where the variables are "linear" and the combination is linear, whereas the adjustment process exhibits threshold effects. Gonzalo and Pittarakis (2006b) take an opposite direction where the cointegrating relationship exhibits threshold effects whereas the adjustment is linear[5]. That is, y_t is I(1) and x_t follows a SETAR process, but there exists a linear combination of them which is stationary. Note that in that case, the notion of integratedness is undefined and there is no corresponding VECM representation (Gonzalo and Pittarakis 2006a).

Empirical applications Since the seminal work of Balke and Fomby, threshold cointegration has become widely applied in various contexts. The law of one price (LOP) and the associated purchasing power parity (PPP) hypothesis represent maybe the field where the greatest number of studies has been conducted (see for a review on LOP Lo and Zivot, for the PPP, Gouveia and Rodrigues 2004, Heimonen 2006, Bec et al. 2004). Numerous studies on price transmission of agricultural products or other commodities (principally oil) use the threshold cointegration framework. In the field of the term interest theory, threshold cointegration methods have been developed and applied by Enders and Granger (1998), Enders and Siklos (2001), Bec et al. (2008), Krishnakumar and Netto (2009). Other fields include the Fisher effect (Million 2004), financial integration based on comparing local and the US stock markets (Jawadi et al. 2009), exchange rate pass-through (Al-Abri and Goodwin 2009). To my knowledge, however, the use of threshold cointegration remained within the economical literature and no study has been done on other domains.

3 The TAR model: probabilistic structure

The Balke and Fomby approach was based on the use of the SETAR model developed by Chan and Tong associated to cointegration. The SETAR is actually a particular case of the more general TAR model that can be written as:

$$y_t = \begin{cases} \mu_1 + \rho_{1,1} y_{t-1} + \ldots + \rho_{1,p1} y_{t-p1} + \varepsilon_t & \text{if} \quad x_{t-d} \geq \theta_{m-1} \\ \mu_2 + \rho_{2,1} y_{t-1} + \ldots + \rho_{2,p2} y_{t-p2} + \varepsilon_t & \text{if} \quad \theta_{m-1} \geq x_{t-d} \geq \theta_{m-2} \\ \ldots & \text{if} \quad \theta_{\ldots} \geq x_{t-d} \geq \theta_{\ldots} \\ \mu_m + \rho_{m,1} y_{t-1} + \ldots + \rho_{m,pm} y_{t-pm} + \varepsilon_t & \text{if} \quad \theta_1 \geq x_{t-d} \end{cases} \tag{5}$$

[4]Current tests work only with a stationary external variable.

[5]For an analogous case in the structural break literature, see Gregory and Hansen (1996)

This model has several parameters:

- m: the number of regimes
- $\mu_1 \dots \mu_m$: the intercepts in each regime
- $p_{j,1} \dots p_{j,m-1}$: the number of lags in regime j
- $\theta_1 \dots \theta_{m-1}$: the thresholds
- d: the delay of the transition variable
- x_{t-d}: the transition variable

When used in the framework of cointegration, BF used a simplified form of 5 by taking lagged values as the transition variable (i.e. setting $x_{t-d} = y_{t-d}$), which leads to she called self-exciting threshold autoregressive model (SETAR), that they nevertheless called simply TAR. They furthermore set the delay value to $d = 1$, as it then corresponds to the delay of the error-correction term. Note that some authors don't take as transition variable the deviations from equilibrium but rather an external variable[6], i.e. they use a TAR model.

The theoretically unlimited number of regimes is usually restricted to 2 or 3 in empirical studies. Hence, the simplified model takes the following form:

$$y_t = \begin{cases} \mu_L + \rho_{L,1} y_{t-1} + \ldots + \rho_{L,pL} y_{t-pL} + \varepsilon_t & \text{if} \quad y_{t-1} \leq \theta_L \\ \mu_M + \rho_{M,1} y_{t-1} + \ldots + \rho_{M,pM} y_{t-pM} + \varepsilon_t & \text{if} \quad \theta_L \leq y_{t-1} \leq \theta_H \\ \mu_H + \rho_{H,1} y_{t-1} + \ldots + \rho_{H,pH} y_{t-pH} + \varepsilon_t & \text{if} \quad \theta_H \leq y_{t-1} \end{cases} \tag{6}$$

Sufficient and necessary conditions for stationarity of model 6 in case of i.i.d ε_t were derived by Chan et al. (1985) in the case when only one lag is present in each regime[7]. The whole process is stationary if and only if one of the following conditions holds:

1. $\rho^L < 1, \rho^H < 1, \quad$ and $\rho^L \rho^H < 1$
2. $\rho_L < 1, \rho_H = 1, \quad$ and $\mu_H < 0$
3. $\rho_H < 1, \rho_L = 1, \quad$ and $\mu_L > 0$
4. $\rho_H = \rho_L = 1, \quad$ and $\mu_H < 0 < \mu_L$
5. $\rho_H \rho_L = 1, \rho_L < 0, \quad$ and $\mu + \rho_H \mu_L$

Interestingly, the values of the coefficients in the inner regime do not appear in these conditions. Thus, a unit root in the inner regime won't affect the stationarity of the whole process. Note also that the condition for the AR(1) process $|\rho| < 1$ is relaxed as the autoregressive coefficients have only to be strictly inferior to 1.

[6] Or the differences of a single variable included in the process, as in Krishnakumar and Netto (2009).

[7] Chan et al. actually prove this for the general case with m regimes. In that case ρ_L is to be replaced by ρ_1 and ρ_H by ρ_{m-1}

Condition (1) corresponds to a case where the stationarity of the whole process is due to the stationarity of the outer regimes. It corresponds to the previously described case where adjustment occurs only after some threshold has been reached. This is the case being mostly referred and investigated in the threshold cointegration literature. That condition has been shown to hold (de Jong 2009) under the weaker condition of weakly dependent innovations.

Conditions (2) and (3) are less restrictive as they allow the presence of a unit root in an outer regime. The process is though stationary provided the drift in the unit root regime pushes towards the stationary regime.

Conditions (4) is still less restrictive as then the outer regimes can have both unit roots, but the fact that the drift parameters are of opposed signs ensures that the process will revert to its mean. In one sense, a process driven by condition (4) could correspond to a model of adjustment, as once the inaction band is overtaken, strong mean reversion occurs. See the discussion in section 6.1.

Condition (5) does not correspond in our mind to any clear and intuitive process and is not discussed.

Higher order lag polynomial Sufficient and necessary conditions for a SETAR process with more than one lag are still not known. Sufficient conditions have been derived, but those correspond only to condition (1) of the model with one lag. Hence, one may conjecture that weaker conditions allowing for unit roots in regimes such as (2) to (5) may hold. Chan and Tong (1985) established the sufficient condition that $max_{a\leq i\leq m} \sum_{j=1}^{p} |\rho_{ij}| < 1$, Kapetanios and Shin (2006) require stationarity[8] of the outer regimes, whereas Bec et al. (2004) establish weaker conditions, which have a less intuitive interpretation.

4 Estimation and inference

4.1 Estimation and inference in the long-run relationship representation

4.1.1 The one threshold case

Estimation is discussed first in the long-run relationship for the threshold and slope parameters, with first one and then two thresholds. Estimation of the number of lags is then discussed. Second estimation of the threshold with given cointegrating values is done for 1 and 2 thresholds, and then extended to the case where the cointegrating vector has to be estimated.

Notice that model 6 can be written in a usual regression form as:

$$y_t = I_L \left(\mu_L + \rho_{L,1} y_{t-1} + \ldots + \rho_{L,pL} y_{t-pM}\right) + I_M \left(\mu_M + \rho_{M,1} y_{t-1} + \ldots + \rho_{M,pM} y_{t-pM}\right) + I_H \left(\mu_H + \rho_{H,1} y_{t-1} + \ldots + \rho_{H,pH} y_{t-} \right. \tag{7}$$

where the I_a are dummy functions that take either 0 or 1 depending on if $y_{t-1} \in a \quad where a = L, M$ or H:

$$I_a = \begin{cases} 1 & \text{if } y_{t-1} \in a \\ 0 & \text{else} \end{cases}$$

[8]The roots of the lags polynomial having all roots outside the unit circle.

Estimation of the slope parameters $\beta = (\mu_a,\ \rho_{a,i})$ is straightforward in case of a known threshold: it is simply OLS. Note that as the dummy variables are mutually exclusive[9], the subsets regressors are orthogonal and estimation can also be done independently on the subsets.

Estimation of the threshold parameter is not obvious as the dummy variable is a discontinuous function. Hence, to obtain an estimator minimizing the sum of squares or maximizing the log-likelihood, an analytical form can't be derived, nor can usual optimisation algorithms be used, as the objective function is highly erratic.

A solution is obtained through concentration of the objective function. As the slope estimators given a threshold are OLS, one can reduce the problem by concentrating out the minimization problem through $\beta(\theta)$ and the corresponding sum of squares $SSR(\theta)$. The objective function becomes:

$$\hat{\theta} = \arg\min_{\theta} SSR(\theta) \tag{8}$$

Minimization of 8 is done through a grid search: the values of the variable are sorted, a certain percentage of the first and last values is excluded to ensure a minimal number of observations in each regime, the SSR is estimated for each selected value and the one that minimize the SSR is taken as the estimator. This method has received different name in the literature such as *concentrated LS*, *conditional LS*.

This is implemented in package `tsDyn`through the function *selectSETAR*. The range of value to search inside is specified by the argument `trim` specifying the percentage of extreme values to exclude and the argument `th`, which allows to search among fewer observations, search inside an interval or around a value.

R code

```
library(tsDyn)
data(lynx)
grid<-selectSETAR(lynx, m=1, thDelay=0, trim=0.15, criterion="SSR")
```

output

```
Using maximum autoregressive order for low regime: mL = 1
Using maximum autoregressive order for high regime: mH = 1
Searching on 75 possible threshold values within regimes with sufficient ( 15% ) number of observations
Searching on 75 combinations of thresholds (75) and thDelay (1)
```

R code

```
print(grid)
```

output

```
Results of the grid search for 1 threshold
         Conditional on m=  1
   thDelay   th       SSR
1        0 1388 123102676
2        0 1307 123951941
3        0 1475 124388924
```

[9] i.e. an observation is only in one regime at a time

```
4        0 1676 124516444
5        0 1638 124557228
6        0 1426 125281019
7        0 1623 125340740
8        0 1292 125488021
9        0 1000 125590841
10       0  957 125596144
```

Figure 1 shows the output of the grid search and illustrate the erratic behavior of the objective function. As pointed out by Enders (2004), a strong threshold effect will result in a sharp U-shaped grid search.

Once the threshold has been estimated, it can be plugged into the *setar()* function. One can thus obtain the slope estimates and their asymptotic p-values[10] :

R code

```
set<-setar(lynx, m=1, thDelay=0, th=grid$th)
summary(set)
```

output

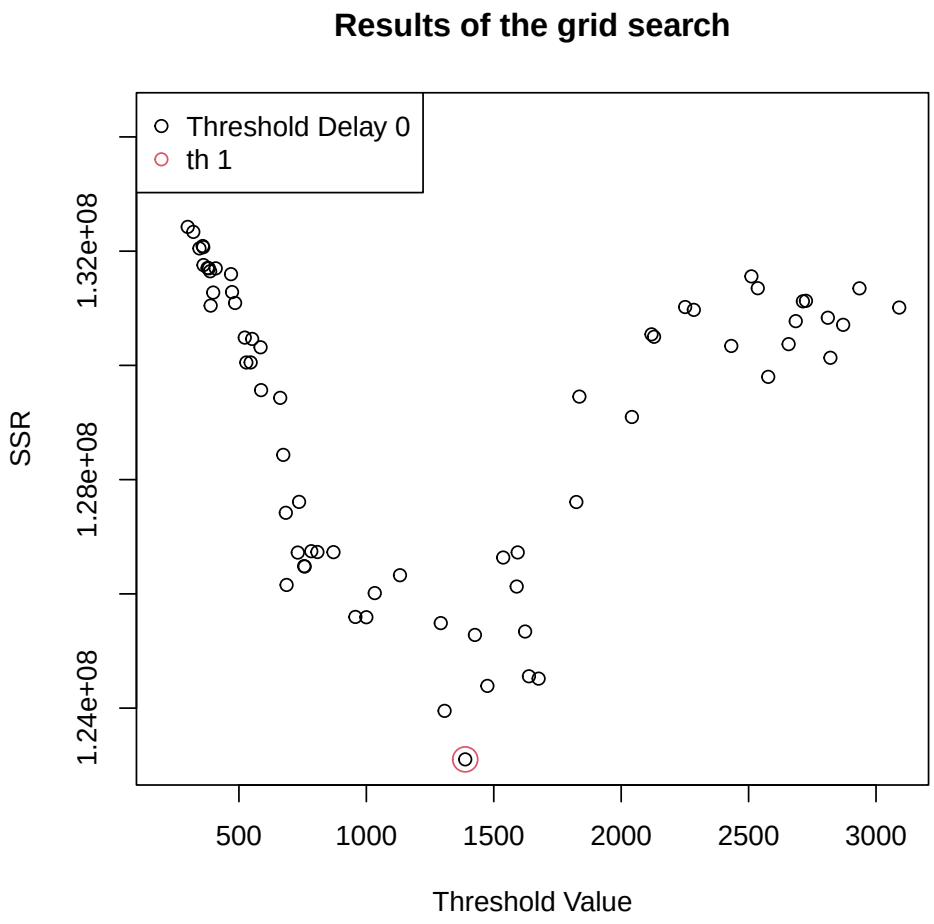
```
Non linear autoregressive model

SETAR model ( 2 regimes)
Coefficients:
Low regime:
    const.L      phiL.1
-150.298119    1.997857

High regime:
    const.H      phiH.1
984.5047382   0.5595309

Threshold:
-Variable: Z(t) = + (1) X(t)
-Value: 1388 (fixed)
Proportion of points in low regime: 59.29%          High regime: 40.71%

Residuals:
      Min        1Q    Median        3Q       Max
-2677.749  -471.918    90.273   327.865  4067.721

Fit:
residuals variance = 1079848,  AIC = 1592, MAPE = 119.8%

Coefficient(s):
```

[10] based on student distribution

Figure 1: Graphical output of the grid search for one threshold

R code

```
plot(grid)
```



```
          Estimate  Std. Error  t value  Pr(>|t|)
const.L -150.29812   220.45996  -0.6817   0.49683
phiL.1     1.99786     0.39437   5.0659 1.651e-06 ***
const.H  984.50474   385.10377   2.5565   0.01194 *
phiH.1     0.55953     0.11439   4.8915 3.442e-06 ***
---
Signif. codes:  0

Threshold
Variable: Z(t) = + (1) X(t)

Value: 1388 (fixed)
```

It produces an object of class `setar` with specific methods such as *print()*, *summary()*, *plot()* and *toLatex()* and inherits from the class *nlar* with general methods for *AIC()*, *BIC()*, *coef()*, *deviance()*, *fitted()*, *logLik()*, residuals(), *MAPE()*, *mse()*, *predict()*.

Note those two steps could have been done directly using *setar()* and without specification of the `th` argument.

4.1.2 The two threshold case

Procedure for two thresholds can be conducted in the same way, and searching on all combinations of θ_L, θ_H to minimize $SSR(\theta_L, \theta_H)$. This is however a n^2dimensional search and may rapidly become cumbersome. A computational shortcut was suggested in BF (1997). The idea is to estimate the threshold in a sequential way: the search is done first in a model with only one threshold. The second threshold is then estimated taking the first as fixed. A few iterations can be conducted, reestimating the first threshold conditional on the second one and viz. Gonzalo and Pittarakis (2002) showed that this algorithm is efficient as the estimator in the first step in the mis-specified model is nevertheless consistent for one of the thresholds. This is a substantial shortcut as it reduces the number of computations from n^2 to $2 \times n$ or $k \times n$ when some iterations are done, practice showing that after 2 or 3 iterations a maximum is reached.

Estimation of the second threshold is done in package `tsDyn`by setting the parameter `nthresh` to 2:

R code

```
selectSETAR(lynx, m=1, thDelay=0, trim=0.15, criterion="SSR", nthresh=2)
```

output

```
Using maximum autoregressive order for low regime: mL = 1
Using maximum autoregressive order for high regime: mH = 1
Using maximum autoregressive order for middle regime: mM = 1
Searching on 75 possible threshold values within regimes with sufficient ( 15% ) number of observations
Searching on 75 combinations of thresholds (75) and thDelay (1)
Result of the one threshold search:
```

```
 -Thresh:  1388         -Delay:  0         - SSR 123102676
Second best: 2577 (conditionnal on th= 1388 and Delay= 0 )          SSR/AIC: 114452658
Second best: 1000 (conditionnal on th= 2577 and Delay= 0 )          SSR/AIC: 113310032
Results of the grid search for 1 threshold
         Conditional on m=  1
  thDelay   th       SSR
1       0 1388 123102676
2       0 1307 123951941
3       0 1475 124388924
4       0 1676 124516444
5       0 1638 124557228

Results of the grid search for 2 thresholds
          Conditional on thDelay =  0  and m = 1
   th1  th2       SSR
1 1000 2577 113310032

Overall best results:
  thDelay       th1       th2       SSR
        0      1000      2577 113310032
With lags:
        -ML: 1
        -MM: 1
        -MH: 1
```

Previous discussion was based on the pure TAR model. When this is applied in the domain of threshold cointegration, the cointegrating vector needs to be estimated. This doesn't seem to be a problem as practically all studies apply a two-step approach, estimating first the cointegrating vector and then estimating the threshold parameters of the residuals from the first step. This could be justified as the estimator of the first step is super-consistent[11]. There are to my knowledge nevertheless no proof nor empirical simulations showing that this sequential approach leads indeed to global optimization over the parameters.

4.1.3 Distribution of the estimator

Properties of the concentrated LS estimator described above were obtained by Chan (1993). He established that the estimator of the threshold, $\hat{\theta}$, was super-convergent, whereas the estimator of the slope coefficients, $\hat{\beta}$, was convergent. He furthermore found that the distribution of $\hat{\theta}$ is a compound Poisson process with nuisance parameters, which can't be computed easily. Superconvergence of $\hat{\theta}$ allows asymptotically to take the estimated value as given and conduct usual inference on the $\hat{\beta}$. Indeed, the distribution of $\hat{\beta}$ is the usual gaussian law and is independent asymptotically of the $\hat{\theta}$. Those results apply when all the coefficients values differ in each regime, the distribution of the whole process being discontinuous. In a certain case when only

[11] Super-consistency refers to the fact that the estimator converge to its true value at rate n instead of usual rate $\sqrt{n}$.

a few variables have regime specific value, the so-called continuous case, Chan and Tsay (1998) established that the threshold estimator converges at the usual rate and is normally distributed, whereas the asymptotic Independence does not hold. However, the continuous model does not seem to have received much use in empirical applications[12]. There remains an uncertainty for me nevertheless if some studies use actually a continuous model but describing it as a discontinuous model.

Note that while in both continuous and discontinuous models the results are known in the one threshold case, there is to my knowledge no study investigating the two thresholds model.

Inference on the threshold parameters A few studies have concentrated on methods to do inference on the threshold parameter. Hansen (2000) makes the assumption that the threshold effect vanishes asymptotically, which enables him to derive the distribution of the threshold parameter and to provide critical values for the likelihood ratio test of $\theta = \theta_0$. Confidence intervals can then be obtained by inverting the log-likelihood ratio: the bound are the values for which the test is rejected. [13]

Gonzalo and Wolf (2005) use a subsampling procedure to obtain confidence intervals for the threshold. Their method has the advantage of providing a test to discriminate between continuous and discontinuous models. Seo and Linton (2007) modify the objective function by replacing the indicator function by a smoothing function[14]. This so called smoothed least square estimator has a smaller rate of convergence but is normally distributed and still independent of the slope parameter estimator. They furthermore establish the validity of a regressor-based bootstrap to obtain small-sample refinements. None of those methods is currently implemented in package `tsDyn`, but the inclusion of Hansen method is under project.

Even if the estimators of the slope parameters are asymptotically normally distributed and independant of the threshold estimator, this may not hold in small sample. Hansen (2000), and Seo and Linton (2006), both suggest methods to take into account the variability of the threshold parameter when building confidence intervals for the slope coefficients. Hansen's method requires lot of computations as it implies to estimate the confidence interval of $\hat{\beta}(\hat{\theta})$ for all θ_i that are included in the confidence interval of $\hat{\theta}$. Using the normality of the smoothed-least square, Seo and Linton (2006) are able to obtain a simpler way to compute the confidence interval for $\hat{\beta}$.

-Carlo Studies Globally, Monte Carlo studies of the estimators in the previous papers (Hansen 2000, Gonzalo and Wolf 2005, Seo and Linton 2006) find that the threshold parameters exhibit a large variability, higher than is predicted by the asymptotical theory: super-convergence of the estimator does not seem to be effective in small samples. Consequently, the slope estimators exhibit a large variability. Without surprise, the authors remark that the variability decrease with the sample size as well as with the effect threshold: the bigger the difference in the parameters in each regime, the better the estimation of the threshold. Gonzalo and Pittarakis (2002) find another interesting factor influencing the variability, namely the number of observations in each regime. While it seems obvious that this will influence the precision in estimating the slope parameters, they show that this affects also the threshold parameter. Indeed, $\hat{\theta}$ is best estimated

[12] Gonzalo and Wolf (2005) discuss a test to differentiate between continuous and discontinuous model.

[13] As the objective function is erratic, there may be intervals inside which the test was rejected for some values and not rejected for others, see graph page 588 in Hansen (2000).

[14] This can be a distribution function as it needs to be bounded between 0 and 1.

when there are an equal number of observations in each regime, precision of the estimator decreasing when only a few observations are present in a regime.

Estimation of the number of lags The estimation of the number of lags can be done be using again the concentration method given above, using as objective function an information criterion (AIC, BIC) rather than the SSR.

$$IC() = n * \log \hat{\sigma}_{\varepsilon}^2 + a(n) * k \tag{9}$$

Where a(n) =2 for the Akaike information criterion (AIC) or a(n)=ln(n) for the bayesian information criterion (BIC)[15].

The parameter are then

$$(\hat{\theta_{1,}}, \hat{\theta_{2,}}, \hat{k_{1,}}, \hat{k_{2,}}, \hat{k_{3,}} = \arg\min IC(\theta_1, \theta_2, k_1, k_2, k_3) \tag{10}$$

This is a considerable extension of the dimension of the grid search, and usually one uses the restriction $k_1 = k_2$.

This is possible in package `tsDyn`by specifying the argument `criterion=AIC` in function *selectSETAR()*:

R code

```
selectSETAR(lynx, m=6, thDelay=0, trim=0.15, criterion="AIC", same.lags=TRUE)
```

output

```
Using maximum autoregressive order for low regime: mL = 6
Using maximum autoregressive order for high regime: mH = 6
Searching on 70 possible threshold values within regimes with sufficient ( 15% ) number of observations
Searching on  420  combinations of thresholds ( 70 ), thDelay ( 1 ) and m ( 6 )
Results of the grid search for 1 threshold
   thDelay m   th      AIC
1        0 2 1388 1528.278
2        0 2 1307 1528.471
3        0 2  808 1529.596
4        0 2 1000 1529.765
5        0 2 1033 1529.830
6        0 2 1292 1529.882
7        0 2 1132 1529.940
8        0 2  957 1530.249
9        0 2  784 1530.425
10       0 2  758 1530.807
```

The argument same.lags restrict the search to have the same number of lags in each regime. Its default value, currently set to `FALSE`[16], search on all combinations of lags, that is, allows to have different lags in each regime.

[15] There are several formulations of those criterions. We took here the formulation as in Franses and van Dijk (2000)

[16] This will probably be changed soon in future version.

4.2 Estimation and inference in the TVECM representation

Estimation Estimation of the threshold and cointegrating parameters could be done in the long-run relationship and those estimates plugged into the TVECM, as the Engle-Granger advocates for the linear case. To my knowledge,the the validity of that method has not been investigated in theoretically. BF mention that the super-convergence of the OLS estimator in the LR (Watson 1987) still hold when the residuals follow a SETAR process under the condition (1).

Rather, Hansen and Seo (2002) and Seo (2009) study estimators directly based on the TVECM. Hansen and Seo derive a maximum-likelihood (ML) estimator, and use a two-dimensional grid for simultaneous estimation of $\hat{\theta}$ and $\hat{\gamma}$. This two-dimensionality can't be avoided as the parameters can't be expressed as functions each of the other one: for each cointegrating value the ECT will be different. For θ, the grid is restricted to the existing values of the ECT, with exclusion of the upper and lower ranges. For the cointegrating value, HS suggest to conduct the search based on a confidence interval obtained in the linear model. When the two values are give, the slope and speed adjustment parameters can be concentrated out and the estimator is simply OLS (though HS depict it as MLE, it is only MLE as starting values for the algorithm are based on the linear MLE estimate). This method can be done in a simple bivariate model without intercept in the cointegrated relationship, but becomes intractable with more than two cointegrating relationships.

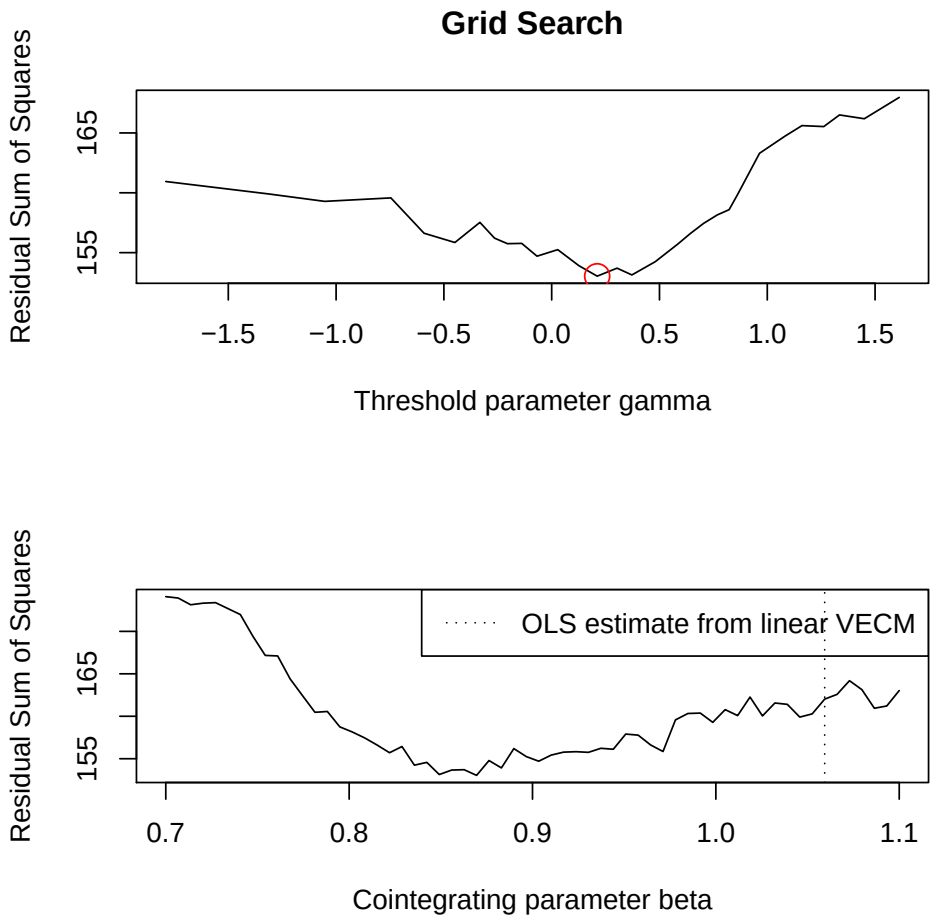
Note that in what I called the cointegration with threshold effect framework, where an external variable rather than the ECT is taken as transition variable, estimation is highly simplified as the interdependency between the ECT term and the threshold variable is ruled out. Estimation of multivariate VECM with many cointegrating relationships is then feasible, the grid search being conduced only over the threshold parameter space (Krishnakumar and Netto 2009).

Inference While Hansen and Seo (2002) suggested an estimator for the multivariate case, they only conjectured its consistency. Interesting results can be found in Seo (2009) concerning proprieties of the LS estimator. Seo shows that LS estimators of both the threshold and cointegrating values are super convergent, the estimator β converging at a faster rate than in linear model, at $n^{\frac{3}{2}}$instead of n. Similarly as in his previous work in the univariate case (Seo and Linton 2007), Seo considers a smoothed-LS estimator and finds that is converging at a slower rate but then normally distributed, allowing to obtain confidence intervals.

Implementation in R The function *TVECM()* in package `tsDyn`allows to estimate a bivariate TVECM with two or three regimes with the OLS like estimator. It should be emphasized here that in my view there is no difference, except in the starting value, between the OLS and MLE estimator, as conditional on the threshold and the cointegrating value, the MLE estimator is simply LS. The model can be specified either with a `constant` a `trend`, or `none`, (arg `include`) and the lags can be regime specific or not (arg `common`).

Procedure for the *TVECM()* differ from that of *setar()* as there is no corresponding *selectSETAR()* function. As the search is two dimensional and the cointegrating parameter take continuous values, it can be easily cumbersome and different options to restrict the search are given with arguments `ngridBeta`, `ngrid`,

Figure 2: Results of the two-dimensional grid search for a TVECM



`Th`, `gamma1`, `gamma2`, `beta`[17].

R code

```
data(zeroyld)
tvecm<-TVECM(zeroyld, nthresh=2,lag=1, ngridBeta=60, ngridTh=30, plot=TRUE,trim=0.05, beta=list(int=c(0.7, 1.1)))
```

It produces an object of class *TVECM()* with specific methods such as *print()*, *summary()* and *toLatex()* and inherits from the class `nlVar` with general methods for *AIC()*, *BIC()*, *coef()*, *deviance()*, *fitted()*, *logLik()*, *residuals()*.

Note that a plot of the search is given automatically as this has proved in practice to be a useful tool, experience showing that the confidence interval for the cointegrating values are too small and hence only a local minimum is obtained, which can be easily detected with the plot.

[17]Name of this argument will probably be modified in further version of the package.

5 Testing

Testing for threshold cointegration is particularly difficult as it involves two aspects: the presence of cointegration and that of non-linearity. Hence, one may have four different cases:

- Cointegration and threshold effects
- Cointegration and no threshold effects
- No cointegration and no threshold effects
- No cointegration and threshold effects

Hence, a test with threshold cointegration may have as null hypothesis either cointegration or no cointegration. This distinction is of major importance as this implies a different distribution under the null. The distribution is also different whenever the test is done based on the LR or the VECM representation[18]. Some of the tests also allow to estimate the cointegrating vector, whereas the majority requires pre-specified ones. Finally, the number of regime differ in the different specifications, some taking two, some three regimes, or symmetric outer regimes. To my knowledge, only one test (Hansen 1999) is able to determine the number of regimes, through a test of one against two thresholds. As a result, there exist many different tests for all the possible cases.

The approach advocated by BF was to conduct a two-step analysis in the LR with pre-specified cointegrating value: testing first for cointegration, and if tests indicate presence of cointegration to test for threshold effects. Nevertheless, this approach may suffer of low power when the true model contains threshold effects and the first step is conduced using tests with a linear specification. Indeed, several studies showed that conventional unit root tests had very low power when the alternative was a stationary SETAR (Pippenger and Goering 2000). Indeed, many studies found that the LOP did not hold, the unit root being not rejected, contrary to many economic arguments in favor of its stationarity.

Taylor (2001) advocated that the failure of tests to reject the unit root for the case of the LOP was due to the use of test which assume linear adjustment. Use of more appropriate tests was indeed able to confirm the LOP. He showed through theoretical and simulation-based arguments that indeed linear tests were biased towards non-rejection of stationarity.

Hence, the procedure should be to do, as in BF, a two-step approach, using first linear tests of cointegration. If linear cointegration is not rejected, tests for threshold cointegration with linear under H_0should be used. Failure of cointegration in the first step should lead to the use of tests with no cointegration under $H_0$and threshold cointegration under the alternative. The second case is particularly interesting, as it illustrates how threshold cointegration is a broader concept that involves linear cointegration as a specific case.

5.1 The problem of the unidentified parameter

A problem for the statistical testing procedure arises when the threshold parameter needs to be estimated. In case of a known threshold parameter, a likelihood-ratio test for the null of no threshold effects (testing

[18] Actually a VAR if the null is no cointegration.

actually equality of the coefficients in each regime) can be formed and has the usual χ^2 distribution (Chan and Tong 1990). But when it is unknown, which is typically the case in practice[19], the distribution of the test is then non-standard as it entails a parameter that is not identified under the null, the so-called Davies problem (1977, 1987).

Solutions for that problem (Andrews and Ploberger 1994) involve usually applying the test statistic for a wide range of possible threshold values, and then aggregating those results. One of the solution encountered is to average all the values, either by using a simple mean or an exponential average. Another solution is to use a supremum statistic, that is the value for which the test is most favorably rejected. This may be seen as an endogeneity bias, but it is not as long as appropriate asymptotical tools are used, that take into account this variability of the test. For a discussion on that question in the similar field of structural break, see Perron (1989) and Andrews and Zivot (1992).

As the test are applied on a range of values, the question of the selection of that range arises. A typical approach is to sort the threshold values in ascending order and exclude a certain percentage of the lowest and highest values. There is no clear rule on the choice of this percentage, but it should not be too small as Andrews and Ploberger (1994) show that setting it too low result in a considerable size distortion. Other approaches as in Bec et al. (2008) is to construct a different grid under the null and under the alternative, using the ADF unit root test for the pre-testing.

The sup-test procedure looks really similar to the estimation procedure as both rely on the use of a sorted grid, with exclusion of some extreme values. The parameter selected nevertheless is the same only in the case of an homoscedastic Wald (or Fisher) test. Indeed, the threshold parameter minimizing the SSR need not be the same of that one maximizing a LM statistic.

Another interesting approach is provided in Altissimo and Corradi (2002) who derive bounds for Wald and LM type tests. Contrary to the usual approach consisting in deriving the asymptotical distribution of the tests and obtaining critical values, they simply show that one may apply a functional to the test that is bounded.

The decision rule from their bound is easy as the model under the null (alternative) should be chosen when the bound is below (above) one. They show that this procedure leads to type I and type II errors approaching zero asymptotically. This result is of great importance as it allows to reduce substantially the number of computation, as critical values don't need to be tabulated.

5.2 Cointegration vs. threshold cointegration tests

5.2.1 Test based on the long-run relationship

As discussed above, the idea for the testing procedure if to test first for cointegration and in the case when cointegration is not rejected, test for threshold cointegration, taking cointegration as a null. In our view, implicetly assumed in that methodology is that the threshold model will be also stationary. Whereas the fact that a unit root may appear in a three-regime SETAR model, the question is never asked in a two regimes-model. This is an important gap as indeed splitting the sample may create a unit root in one of the regime. This is the case indeed in Hansen (1999), who did not seem to note it.

[19] unless maybe when one imposes a threshold of zero

In package `tsDyn`, a minimal test is done automatically computing whether the roots of the polynomials don't have values equal or lower to one. Hence, one obtains an automatical warning with data from Hansen (1999):

R code

```
data(IIPUs)
set<-setar(IIPUs, m=16, thDelay=5, th=0.23)
```

Hansen (1996) derived the asymptotic properties of the sup-LM test for a SETAR model with one unknown threshold. The test follows a complicated empirical distribution process with nuisance parameters and hence critical values for a general case can't be tabulated. Hansen nevertheless shows a simulation procedure which allows to generate asymptotic p-values. In this procedure, heteroskedasticity can also be taken into account by slight modifications. In a later article, Hansen (1999) studies an alternative way to obtain the p-values through a residual bootstrap, whose validity is nevertheless not established but only conjectured. More interestingly, the author develops an extension of the testing procedure to test against two threshold, and to determinate the number of thresholds by testing the null of one threshold against two.

This test is available in `tsDyn`with the function *setarTest()*, for which the homoskedastic bootstrap have been implemented. It takes as argument `nboot` the number of bootstrap replications and `test`="`1vs`" (1 regimes against 2) or "`2vs3`" (2 regimes against 3).

R code

```
Hansen.Test<-setarTest(lynx, m=1, nboot=1000)
```

Available methods are *print()*, *summary()* and *plot()*, as well as an *extendBoot()* function to run new bootstrap replications and merge the result with the old ones. This can be useful for a preliminary test and to check how the result is influenced by new runs.

Another type of test has been suggested by Petrucelli and Davies (1986) and Tsay (1989). By transforming the specification into an arranged autoregression, Tsay reformulates the problem into a structural change test. With a test of stability of recursive residual to detect structural change, the problem of the unidentified parameter under H_0 is avoided and hence the test follows a simple χ^2distribution. This test has been implemented in R but not included in version 0.7 as the result differ sometimes drastically from those in the paper[20]

BF suggested to extend the approach of Tsay using several other structural change tests, using techniques as in Hansen (1996). Appropriateness of this method has been nevertheless discussed by Hansen (2000) as the the ordering of the variable for the arranged autoregression may induce a trend, in case the structural tests are not consistent.

Criterion based approaches Differing from a pure testing procedure, model selection procedures based on information criteria (IC) have gained much interest in the literature and their use has been sometimes advocated rather than formal testing procedure (see for example Lètkephol2007). This is the case in the

[20]actually as well as from those also differing in the GAUSS procedure distributed by Lo and Zivot (2002).

well-known selection of lags in time series models, but also for estimating the cointegrating rank (Gonzalo and Pittarakis 1998, Cheng and Phillips2009).

This has been also applied for the determination of the number of regimes in a SETAR model by Gonzalo and Pittarakis (2002). They show indeed that this works well in practice using a modified BIC. This result is of great interest in practice as it avoids the use of bootstrap replications and hence significantly diminishes the number of computations required. The authors remark in simulations studies that the AIC has big type one error compared to other BIC, while it has a smaller type II error.

This is easily implemented in package `tsDyn`with the generic function *AIC()* and the similar *BIC()*. Furthermore, with the argument `k`, practically any penalty term can be used. Using the example of Hansen (1999):

R code

```
sun<-(sqrt(sunspot.year+1)-1)*2
lin<-linear(sun, m=11)
set1<-setar(sun, m=11, th=7.4, thDelay=1, nested=TRUE)
set2<-setar(sun, m=11, th=c(5.3,8),nthresh=2, thDelay=1, nested=TRUE)
matrix(c(AIC(lin),AIC(set1),AIC(set2),BIC(lin),BIC(set1),BIC(set2)),ncol=2,dimnames=list(c("lin","set1", "set2"),c
```

output

```
          AIC      BIC
lin  419.3361 463.3332
set1 381.2476 472.9083
set2 362.1827 501.5069
```

As mentioned above, one could also use bounds derived by Altissimo and Corradi (2002). The authors indeed investigate proprieties of their bound and find that is has considerable size distorsion but excellent power for the alternative they choose.

This bound has not been implemented in `tsDyn`as results were different compared to other studies (see Galvao 2006) and there is no comparison of the results with other testing procedures.

5.2.2 Test based on the TVECM representation

Hansen and Seo (2002)suggest a sup-LM test of a linear VECM against a threshold VECM with two regimes. In the case of unknown cointegrating vector, the search for the sup-LM maximal value can be reasonably done only for the case of a bivariate TVECM.

$$\begin{bmatrix}\Delta X_t \\ \Delta Y_t\end{bmatrix} = + \begin{cases} \begin{bmatrix}c^{XL} \\ c^{YL}\end{bmatrix} + \begin{bmatrix}a_{XL} \\ a_{YL}\end{bmatrix} ECT_{L,t-1} + B_{1L}\begin{bmatrix}\Delta X_{t-1} \\ \Delta Y_{t-1}\end{bmatrix} + \ldots + B_{pL}\begin{bmatrix}\Delta X_{t-p} \\ \Delta Y_{t-p}\end{bmatrix} \\ \begin{bmatrix}c^{XH} \\ c^{YH}\end{bmatrix} + \begin{bmatrix}a_{XH} \\ a_{YH}\end{bmatrix} ECT_{H,t-1} + B_{1H}\begin{bmatrix}\Delta X_{t-1} \\ \Delta Y_{t-1}\end{bmatrix} + \ldots + B_{pH}\begin{bmatrix}\Delta X_{t-p} \\ \Delta Y_{t-p}\end{bmatrix} \end{cases} \tag{11}$$

Collecting the various parameters into $A_L = C_L, a_L, B_L$and similarly for A_H, the H_0 of a linear model $A_H = A_L$is rejected when $A_H \neq A_L$. The distribution of the sup-LM test is found to be the same as in the univariate case as in Hansen (1996). This distribution can't be tabulated due to the presence of

nuisance parameters and hence the authors suggest two bootstrap approaches, with either a fixed-regressor or a residual bootstrap. While in the paper the sup-LM tests is conditional on both the cointegrating and threshold value, the implementation of this test done by the authors takes the cointegrating vector as given, based on the value estimated from the linear VECM.

This is available in package `tsDyn`using the function *TVECM.HStest()*.

5.3 No cointegration vs. threshold cointegration tests

Numerous tests for the null of no cointegration may be used, either from the L-R or the TVECM representation. They generally suffer from two major drawbacks which merit on my opinion more attention.

The first is that unit roots tests with a stationary SETAR as alternative which developed recently may be used in the case of a known cointegrating vector, in analogy to the linear case (see section 1). This has curiously not been discussed by their authors. Formal test which allow to estimate the beta have not been derived in the L-R nor in TVECM form, and hence constrain the threshold cointegration field to analyze only cases where the cointegrated values are meant to be known. Even if it is a strong restriction, it is still interesting, since there are many applications where theory predicts a particular cointegrated vector, as Horvath and Watson (1995) claim:

> Economic models often imply that variables are cointegrated with simple and known cointegrating vectors. Examples include the neoclassical growth model, which implies that income, consumption, investment, and the capital stock will grow in a balanced way, [...]. Asset pricing models with stable risk premia imply corresponding stable differences in spot and forward prices, long- and short-term interest rates, and the logarithms of stock prices and dividends. Most theories of international trade imply long-run purchasing power parity, so that long-run movements in nominal exchange rates are matched by countries relative price levels. Certain monetarist propositions are centered around the stability of velocity, implying cointegration among the logarithms of money, prices, and income.

A second drawback of the test with the null of no cointegration is that the alternative stationary model always take the form of the condition (1). Hence, the presence of a unit root is considered as non-stationarity of the series. We saw nevertheless above that a SETAR process may still be stationary even with a unit root in a regime. Henceforth, the non-rejection towards stationarity is not in itself a sign that the series is indeed non-stationary. It is only a first step indicating that condition (1) of stationarity does not hold, but this does not mean that other conditions may not hold and the process hence be stationary. However, those further investigations are not possible as there are to my knowledge no tests that take as alternative hypothesis other conditions such as (2) or (3). We may hence conjecture that many of the series described as non-stationary in the literature may well be SETAR-stationary.

5.3.1 Tests based on the long-run relationship

Taking no cointegration as null hypothesis has the implication that under the null, the series (which is also, remember, the transition variable) is non-stationary, which affects the distribution of the tests. Anal-

ogously as in the linear case, when the cointegrating vector is known, usual unit root tests can be used, whereas estimation of the cointegrating vector affect the distribution of the test, which require use of different tests/critical values. Hence, when the cointegrating vector is known, unit root tests with a stationary SETAR as alternative can be used, whereas the case of unknown cointegrating vector needs correction.

Known threshold

Bec, Ben Salem and Carrasco (BBC) Bec, Ben Salem and Carrasco (2004) (referred as to BBC), test for unit root against a symmetric three regime SETAR model. The model specification is very general as intercepts as well as lags are included in each regime, and hence corresponds to the model in 7.

$$\Delta y_t = I_L \left(\mu_L + \rho_L y_{t-1} + \sum \Delta\gamma_{L,i} y_{t-i}\right) I_M \left(\mu_M + \rho_M y_{t-1} + \sum \Delta\gamma_{M,i} y_{t-i}\right) I_H \left(\mu_H + \rho_H y_{t-1} + \sum \Delta\gamma_{H,i} y_{t-i}\right) + \varepsilon_t \tag{12}$$

where $I_L = I_{\{y_{t-1} \leq -\theta\}}$, $I_H = I_{\{y_{t-1} > \theta\}}$ and $I_M = I_{\{-\theta \leq y_{t-1} \leq \theta\}}$

The null hypothesis of unit root is H_0: $\rho_L = \rho_H = \rho_M = 0$ with the alternative H_A: $\rho_L < 1$, $\rho_H < 0$ and $\rho_M \leq 1$. That is, a unit root is allowed in the middle regime, but an explosive behavior is ruled out. BBC find that the distribution of sup-Wald, sup-LM and sup-LR are free of nuisance parameters and provide critical values. The authors suggest that the extension to a SETAR with non symmetric thresholds should not lead to further complications.

Kapetanios and Shin (KS) Kapetanios and Shin (2006) uses as alternative a three-regime model with a unit root in the inner middle which is meant to be more consistent with the concept of a band without adjustment. As then coefficients in the middle regime do not need to be estimated, the test is meant to have better power when the true model is indeed a model with a unit root in the inner regime. They allow the possibility to add lags common to all the regimes.

$$\Delta y_t = \mu_L + \rho_L y_{t-1} + \mu_H + \rho_H y_{t-1} + \sum_{i=1}^{p} \gamma_i \Delta y_{t-i} + \varepsilon_t \tag{13}$$

The null hypothesis of unit root is H_0: $\rho_L = \rho_H = 0$ with the alternative H_A: $\rho_L < 1$, $\rho_H < 0$. The grid for the thresholds is selected such that the probability of being in the middle regime decreases as the sample size increases, converging to zero asymptotically. Under this specification, the authors derive statistics (the sup-Wald, exp Wald and ave Wald) which are obtain nuisance parameter free and provide critical values.

Bec, Guay and Guerre (BGG) Under a similar model such as in BBC, Bec, Guay and Guerre (2008) (hereafter BGG) concentrate on the selection of the grid to obtain a consistent test diverging under H_A. The idea is that if H_0 is true, the grid should be small as to ensure a good size, whereas if H_A holds, the grid should be as large as possible to ensure power. Hence, the width of the grid is selected depending on a pre-test based on the AD test.

Seo Seo (2008) derive a test with a two-regime SETAR as alternative, allowing for non linear and serial correlated errors. Under those weaker assumptions, the asymptotical distribution of the sup-Wald selecting

the values as in KS depends on nuisance parameters and critical values can't be tabulated. Seo hence suggests a residual based block bootstrap, shown to be asymptotically consistent. Extension of the bootstrap for a three regime model is meant to be easy.

Monte Carlo comparisons Maki (2009) provides a Monte Carlo simulation for the size and power of the BBC (using sup-Wald) and KS (sup-Wald, ave and exp-Wald) tests, along with the traditional ADF. Size of the test is definitely better for the ADF test, with the ave-Wald being close, the sup- and exp-Wald of KS showing size distorsion, while the sup-wald of BBC is seen to be too conservative. Power of the tests is investigated based on an alternative model with three regimes, a symmetric threshold, a unit root in the middle regime and no lags. Different values of the thresholds are tested. With a threshold value of zero (i.e. no thresholds effects), the ADF test has without surprise the best power. More surprisingly, this is still the case with small thresholds, with values 1 and 2, the ADF test having the best power for models with an inner regime counting 40% of the observations. When the thresholds increase (and hence the number of observations in the unit root regime), power of the ADF decreases consequently. This is also the case for the KS test, as it is based on a asymptotically degenerated threshold, whereas there is no clear effect on the BBC. As the KS test does not estimate the inner regime whereas the BBC does, the power of the KS is much higher, because the true process has indeed a unit root.

When the mean-reversion of the outer regime is increased (which has the effect to have more observations in the inner regime), all the tests have power near to 1 unless the threshold are high (value of 6), where power of ADF test falls. The BBC test has in those cases a much better power than the KS, as the latter is based on a diminishing threshold effect.

Implementation in R BBC and KS have been implemented in package `tsDyn`. They are nevertheless for now in an experimental version and may contain errors. Practitioners should use them with care as the results could not be compared to those of the authors as the data sets are not publicly available.

5.3.2 Unknown cointegrating values

Two early tests deserve here mention. Enders and Granger (1998) first provided an empirical framework to deal with unit root tests having a two-regime SETAR as alternative. They tabulated critical values for a test when the threshold is known. Critical values with unknown threshold were given later by Enders (2001). Enders and Siklos (2001) adopt similar approach but relying on the cointegration framework and allowing to estimate the cointegrating values. They hence provide larger critical values from those of Enders and Granger (1998).

This is to my knowledge the only test which allows to work with an unknown cointegrating vector. It is not sure nevertheless whether it more appropriate to use this one rather that the more formal unit root tests of BBC and KS, as no distribution theory is given in the former.

5.3.3 Test based on the TVECM representation

Seo (2006) is to my knowledge the only one to discuss a test of no cointegration against threshold cointegration based on the VECM model. His framework is a TVECM model with the ECT splitted into three regime, the middle one being not adjusted and not taken into account, and lags common to all regimes:

$$\Delta X_t = \mu + \begin{cases} a_L ECT_{L,t-1} \\ a_H ECT_{H,t-1} \end{cases} + C_1 \Delta X_{t-1} + \ldots + C_p \Delta X_{t-p} + \varepsilon_t \tag{14}$$

The idea of the test is based on results as in Horvath and Watson (1995), who show that when the cointegrating vector is known, a test of cointegration can be simply done on testing whether coefficients from the ECT are significant. In the TVECM framework, the null hypothesis of no cointegration becomes: H_0: $a_L = a_H = 0$ and the alternative that either a_L or a_H is different from zero. The sup-Wald test suggested does not depend on nuisance parameter and critical values can be obtained. As the asymptotical distribution is seen to perform badly in small samples, Seo provide a residual based bootstrap and shows its asymptotic consistency.

This test is available in package `tsDyn`as *TVECM.SeoTest()*.

R code

```
data(zeroyld)
dat<-zeroyld
testSeo<-TVECM.SeoTest(dat, lag=1, beta=1, nboot=1000)
summary(testSeo)
```

It requires the argument `beta` for the cointegrating value and `nboot` for the number of bootstrap replications. The methods *print()*, *summary()* and *plot()* are available for objects issued by *TVECM.SeoTest()*. As the model specification is done for two thresholds, a two-dimensional grid search has been implemented. This is definitely very slow and a single test may take a few hours [21].

5.4 Conclusion for the test

I reviewed here some of the most popular and applied tests in the literature. This is definitely not exhaustive and there exist many different tests, especially for the univariate case. Despite of this great amount of different tests and model specification, some points

6 Interpretation

Once threshold cointegration have been indicated by the different tests and estimation made, there remain interesting questions on the interpretation of the model obtained. A first one concerns the type of adjustment and the presence of an attractor. A second one concerns stability of the system and its reactions to exogeneous shocks.

[21] As this is a sup-Wald test, for which the best thresholds pair is also maximizing the OLS criterion, one may think that the conditional search could be applied. This is currently under implementation.

6.1 Types of adjustment

As was presented earlier, a SETAR model may be stationary under diverse conditions. Nevertheless, almost the only one to have been investigated empirically, and for which there exist some tests, is that described by the condition (1). Within this type, there are nevertheless different degrees of adjustment, where adjustment pushes back either to an equilibrium (EQ-SETAR) or to a band (BAND-SETAR). This distinction is best shown in a case of perfectly symmetric SETAR with three regimes but same (of opposed signs) thresholds, same outer coefficients, no lags and a random-walk without drift in the inner regime. The EQ-SETAR is then:

$$y_t = \begin{cases} \rho y_{t-1} + \varepsilon_t & \text{if } y_{t-1} > \theta \\ y_{t-1} + \varepsilon_t & \text{if } -\theta < y_{t-1} < \theta \\ \rho y_{t-1} + \varepsilon_t & \text{if } y_{t-1} < -\theta \end{cases} \tag{15}$$

Interpretation of such model is done remembering that the conditional mean in each regime is given by the mean of an AR(1) process, that is $\frac{\mu}{1-\rho}$. So in, 15, the conditional mean in each regime is 0 and hence one may think that 0 is the mean of the global process as well as its equilibrium, hence its name of equilibrium-SETAR. A model with a different type of adjustment, the BAND-SETAR, is given by:

$$y_t = \begin{cases} \theta(1-\rho) + \rho y_{t-1} + \varepsilon_t & \text{if } y_{t-1} > \theta \\ y_{t-1} + \varepsilon_t & \text{if } -\theta < y_{t-1} < \theta \\ -\theta(1-\rho) + \rho y_{t-1} + \varepsilon_t & \text{if } y_{t-1} < -\theta \end{cases} \tag{16}$$

Here, the adjustment process is pushing the deviations exceeding the threshold back to the edge of the band, defined by the space between the two thresholds. Indeed, the conditional mean of the outer regimes are equal to the threshold $\left(\frac{\theta(1-\rho)}{(1-\rho)} = \theta\right)$, whereas the conditional mean of the inner regime is equal to 0. Therefore, adjustment does not push back to a certain equilibrium point such as in a EQ-SETAR but just pull within a certain band. The more complete BAND-SETAR nests the EQ-SETAR so determining which model describe best the data can be easily made.

Whereas the adjustment seems to be faster in a EQ-SETAR than in a BAND-SETAR, it could be that a SETAR described by condition (4) (denoted by returning-drift, RD-SETAR, by BF) has a faster adjustment. Effectively, the outer regimes are unit root process with drift that may have much faster dynamics than simple AR processes. Despite of this potentially interesting feature, RD-SETAR models don't seem to have been used much in the literature, probably because of the unavailability of tests as we discussed below and also because of the more complicated distribution of the parameters.

Note that the difficulty of the comparison of adjustment described by conditions (1) to (4) comes partly from the fact that first the computation of the mean of a SETAR model is difficult and requires numerical methods, and second that the relevance itself of the concept of mean for non-linear model may be questioned, due for example of the potential bimodality of the distribution of the process. Concepts such as attractors and equilibria may be more adequate (see Tong 1990), with the drawback nevertheless that those only be described case to case. By extension, the term itself of mean-reversion in the SETAR framework may be

misleading. This is the case in the process 1 where mean-reversion adjustment appears as a more restrictive condition than stationarity: taking the case where $\mu_H > \theta_H$ and $\mu_L < \theta_L$along with $|\rho_L| < 1$ and $|\rho_H| < 1$ ensures stationarity but not mean-reversion to the inaction band: once in the outer regime, the process may remain there.

Whereas the seminal paper of BF was based on models with three regimes (in some cases the outer regimes are symmetric which may appear as a two-regime specification), other studies (Enders and Granger 1998) focused on two-regimes model. In our view, two-regimes models offer interesting insights into asymmetric behaviors, though they have a complicated interpretation. Empirical studies seek indeed to estimate a threshold in a two-regime model rather than imposing its value to zero. Whereas in a three regime model it makes sense to observe a strictly positive and a strictly negative threshold, there are few economical arguments in favor of a two-regime model with a non-zero threshold, where for example positive deviations would behave like small negative deviations but differently from big ones.

6.2 Non linear impulse response functions

Note that the package tsDyn does NOT provide generalised impulse response functions, altough it does provide standard imuplse response functions for the linear VAR/VECM only, building on the function from package *vars/urca*. Several people have been asking for this functionality, but it should not be too difficult to do it with the existing functions, since tsDyn already implements a TVECM.sim() and TVECM.boot() functions. For more informations, see various discussions on the tsDyn mailing lists: `http://groups.google.com/group/tsdyn/t/5c517a94a3a3ab0c`

7 Running the functions on parallel CPUs

A major drawback of the threshold cointegration tools is that those, due to the probem of the unidentified parameter or the need of bootstrap replications, are heavily computer intensive. The test of Seo (2006) takes indeed a very long time to run.

To alleviate these problems, a possibility is to run the functions on parallel, ie. either on a unique computer with multiple-CPUs processor or on more complex computer clusters. Nowadays, it is quite common to find even laptops equiped with processors like Intel Dual-Core, and hence parallel functionalities can be used by practically everyone. Furthermore, it has become quite easy to do it in R thanks to packages like *foreach* which offer a great level of abstraction, requiring the user to do a minimal number of steps to get it running. Indeed, this package will function as a wrapper for other parallel packages, and allows the user to run it either on the protocol `MPI`, `nws` and `pvm`, or using the R internal `socket` system, as well as the *multicore*[22].

Furthermore, parallel computation is quite easy in the context of threshold cointegration as the grid search or the bootstrap replications are independant of each other and can easily be run on different nodes.

Parallel facilities are for now only available for the function *TVECM.HStest()*, through the argument `hpc`

[22]For a review on R facilities for high performance computing, see Schmidberger, Morgan, Eddelbuettel, Yu, Tierney, and Mansmann (2009). There exist also a dedicated R mailing list, as well as a *task view* `http://stat.ethz.ch/CRAN/web/views/HighPerformanceComputing.html`

(standing for high performance computing). When set to `foreach`, the package will run the *foreach* package. It is then up to the user to choose a paralellisation protocol, the current choice being now between *doMC*, *doSNOW* (MPI, pvm and nws, as well as internal R sockets), *doMPI* and *doRedis*. We illustrate here with the easiest package, *multicore*, wich proved to be quite powerful, with the disadvantage nerverthless that it can pose problems when R is used within a GUI.

R code

```
system.time(test1<-TVECM.HStest(dat, lag=1, nboot=200))
library(doMC)
registerDoMC(2) #Number of cores
system.time(test1<-TVECM.HStest(dat, lag=1, nboot=200, hpc="foreach"))
```

Results are quite impressive, as they show that by simply adding a second core the execution time is divided by two, while using 4 cores will divide the time by three, as shown in figure 3[23].

8 Conclusion

In this paper, I showed the interest of threshold cointegration towards traditional cointegration as being a better framework to model real world adjustment process with stickiness and asymmetries. Indeed, a great number of empirical studies applied this model, and an evenly great number of theoretical results have been obtained. I presented also how one can use those developments using the package `tsDyn`, offering a comprehensive framework for analysis and testing that, despite of the great interest in this field, that was until now not available. Using this package, one may conduce a whole analysis, testing for threshold cointegration in different situations and different model specifications, and estimating those models.

Whereas great developments occured since the seminal work in 1997, many questions remain unanswered . The complexity of the SETAR model is actually so high that simple aspects such its distribution or its moments are still only known in restricted cases. Estimation of more than one threshold stills create problems, and actual tests of stationary consider only a small amount of the possible features of a SETAR. There is up to now no framework allowing to test the stationarity with an unknown cointegrating vector, and test discriminating 2 against 3 regimes only work in a restricted case.

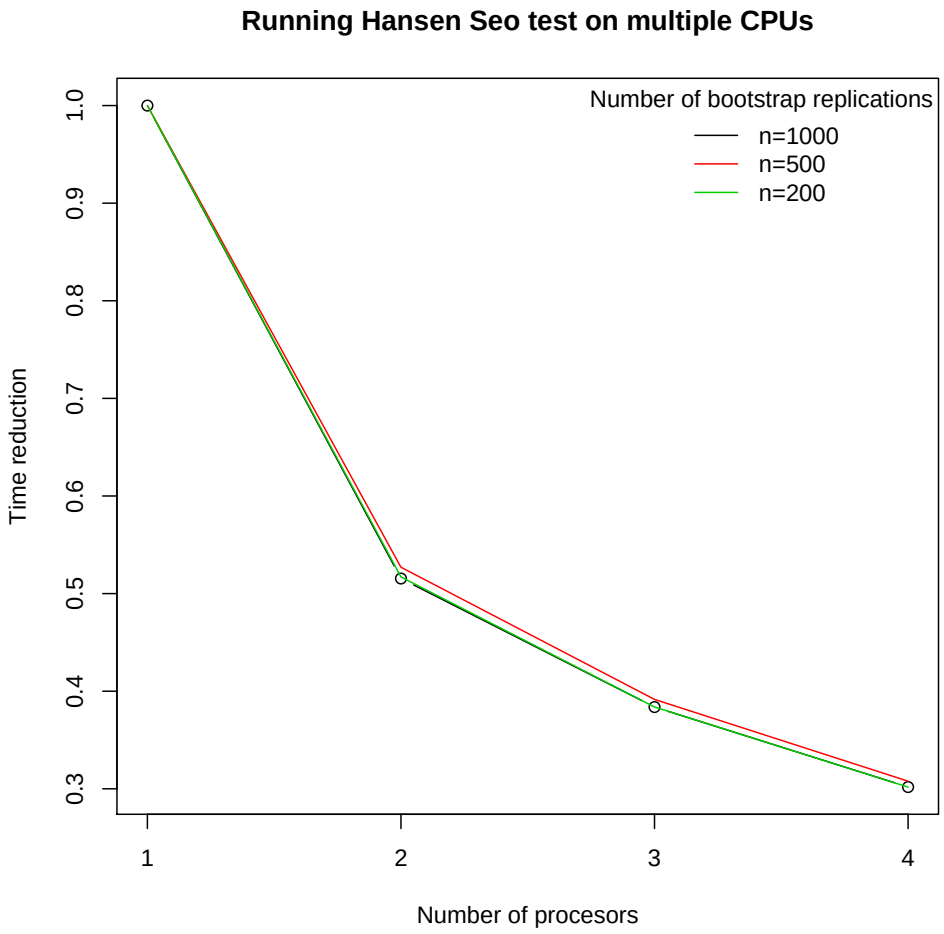
References

Al-Abri, A. S., and B. K. Goodwin (2009): "Re-examining the exchange rate pass-through into import prices using non-linear estimation techniques: Threshold cointegration," *International Review of Economics & Finance*, 18(1), 142–161.

Altissimo, F., and V. Corradi (2002): "Bounds for inference with nuisance parameters present only under the alternative," *Econometrics Journal*, 5(2), 494–519.

[23]It should be mentioned that the relationship between the number of CPUs and the reduction in execution time is decreasing with the number of CPUs: the more the CPUs, the less their additional effect is.

Figure 3: Execution time of function TVECM.HStest using multiple cores



Andrews, D. W. K., and W. Ploberger (1994): "Optimal Tests When a Nuisance Parameter Is Present Only under the Alternative," *Econometrica*, 62(6), 1383–1414.

Balke, N. S., and T. B. Fomby (1997): "Threshold Cointegration," *International Economic Review*, 38(3), 627–45.

Bec, F., A. Guay, and E. Guerre (2008): "Adaptive consistent unit-root tests based on autoregressive threshold model," *Journal of Econometrics*, 142(1), 94–133.

Bec, F., M. B. Salem, and M. Carrasco (2004): "Tests for Unit-Root versus Threshold Specification With an Application to the Purchasing Power Parity Relationship," *Journal of Business & Economic Statistics*, 22, 382–395.

Chan, K., and H. Tong (1985): "On the use of the deterministic Lyapunov function for the ergodicity of stochastic difference equations," *Advances in applied probability*, 17(3), 666–678.

——— (1990): "On Likelihood Ratio Tests for Threshold Autoregression," *Journal of the Royal Statistical Society*, 52(3), 469–476.

Chan, K. S. (1993): "Consistency and limiting distribution of the least squares estimation of a threshold autoregressive model," *The Annals of Statistics*, 21, 520–533.

Chan, K. S., J. Petruccelli, S. Woolford, and H. Tong (1985): "A multiple threshold AR(1)model," *Journal of applied probability*, 22(2), 267–279.

Chan, K. S., and R. S. Tsay (1998): "Limiting properties of the least squares estimator of a continuous threshold autoregressive model," *Biometrika*, 85(2), 413–426.

Cheng, X., and P. C. B. Phillips (2009): "Semiparametric cointegrating rank selection," *Econometrics Journal*, 12(s1), S83–S104.

Damania, R., and B. Z. Yang (1998): "Price Rigidity and Asymmetric Price Adjustment in a Repeated Oligopoly," *Journal of Institutional and Theoretical Economics (JITE)*, 154(4), 659–.

Davies, R. (1977): "Hypothesis testing when a nuisance parameter is present only under the alternative," *Biometrika*, 64, 247–254.

——— (1987): "Hypothesis testing when a nuisance parameter is present only under the alternative," *Biometrika*, 74, 33–43.

de Jong, R. M. (2009): "Nonlinear time series models and weakly dependent innovations," Discussion paper, Department of Economics, Ohio State University.

Dutta, S., M. Bergen, D. Levy, and R. Venable (1999): "Menu costs, posted prices, and multiproduct retailers.," *Journal of Money, Credit, and Banking*, 31(4), 683–703.

Enders, W. (2001): "Improved Critical Values for the Enders-Granger Unit-Root Test," *Applied Economics Letters*, 8(4), 257–61.

——— (2004): *Applied Econometric Time Series*. Wiley, Hoboken, NJ, 2nd ed edn.

Enders, W., and C. Granger (1998): "Unit-root tests and asymmetric adjustment with an example using the term structure of interest rates.," *Journal of Business and Economic Statistics*, 16(3), 304–311.

Enders, W., and P. L. Siklos (2001): "Cointegration and Threshold Adjustment," *Journal of Business & Economic Statistics*, 19(2), 166–76.

Engle, R. F., and C. Granger (1987): "Co-integration and error correction: representation, estimation and testing.," *Econometrica*, 55(2), 251–276.

Franses, P. H., and D. van Dijk (2000): *Nonlinear Time Series Models in Empirical Finance*. Cambridge University Press.

Galvao, A. B. C. (2006): "Structural break threshold VARs for predicting US recessions using the spread," *Journal of Applied Econometrics*, 21(4), 463–487.

Gonzalo, J., and J. Pitarakis (2006a): "Threshold Effects in Multivariate Error Correction Models," in *Palgrave Handbook of Econometrics, Volume 1: Econometric Theory*, ed. by T. C. Mills, and K. Patterson, pp. 578–609. Palgrave MacMillan.

Gonzalo, J., and J.-Y. Pitarakis (1998): "Specification via model selection in vector error correction models," *Economics Letters*, 60(3), 321–328.

——— (2002): "Estimation and model selection based inference in single and multiple threshold models," *Journal of Econometrics*, 110(2), 319 – 352.

——— (2006b): "Threshold Effects in Cointegrating Relationships," *Oxford Bulletin of Economics and Statistics*, 68(s1), 813–833.

Gonzalo, J., and M. Wolf (2005): "Subsampling inference in threshold autoregressive models," *Journal of Econometrics*, 127(2), 201–224.

Gouveia, P., and P. Rodrigues (2004): "Threshold Cointegration and the PPP Hypothesis," *Journal of Applied Statistics*, 31(1), 115–127.

Granger, C., and P. Newbold (1974): "Spurious regressions in econometrics," *Journal of Econometrics*, 2, 111–120.

Granger, C. W. J., and T. H. Lee (1989): "Investigation of Production, Sales and Inventory Relationships Using Multicointegration and Non-Symmetric Error Correction Models," *Journal of Applied Econometrics*, 4, 145–159.

Gregory, A., and B. Hansen (1996): "Test for cointegration in models with regime and trend shifts," *Oxford bulletin of economic and statistics*, 58(3), 555–560.

Hansen, B. E. (1996): "Inference When a Nuisance Parameter Is Not Identified under the Null Hypothesis," *Econometrica*, 64(2), 413–30.

——— (1999): "Testing for Linearity," *Journal of economic Surveys*, 13(5), 551–576.

——— (2000): "Sample Splitting and Threshold Estimation," *Econometrica*, 68(3), 575–604.

Hansen, B. E., and B. Seo (2002): "Testing for two-regime threshold cointegration in vector error-correction models," *Journal of Econometrics*, 110, 293–318.

Heimonen, K. (2006): "Nonlinear adjustment in PPP evidence from threshold cointegration," *Empirical Economics*, 31(2), 479–495.

Horvath, M. T., and M. W. Watson (1995): "Testing for Cointegration When Some of the Cointegrating Vectors are Prespecified," *Econometric Theory*, 11(05), 984–1014.

Jawadi, F., N. Million, and M. E. H. Arouri (2009): "Stock market integration in the Latin American markets: further evidence from nonlinear modeling," *Economics Bulletin*, 29(1), 162–168.

Johansen, S. (1988): "Statistical Analysis of Cointegration Vectors," *Journal of Economic Dynamics and Control*, 12, 231–254.

Johansen, S. (1995): *Likelihood-based inference in cointegrated Vector Autoregresive Models*. Oxford University Press.

Kapetanios, G., and Y. Shin (2006): "Unit root tests in three-regime SETAR models," *Econometrics Journal*, 9(2), 252–278.

Krishnakumar, J., and D. Neto (2009): "Estimation and Testing for the Cointegration Rank in a Threshold Cointegrated System," Discussion paper, Working papers, Econoemtrics department University of Geneva.

Levy, D., M. Bergen, S. Dutta, and R. Venable (1997): "The Magnitude of Menu Costs: Direct Evidence From Large U. S. Supermarket Chains," *The Quarterly Journal of Economics*, 112(3), 791–825.

Lo, M. C., and E. Zivot (2001): "Threshold Cointegration and Nonlinear Adjustment to the Law of One Price," *M acroeconomic Dynamics*, 5, 533–576.

Lütkepohl, H. (2007): *New Introduction to Multiple Time Series Analysis*. Springer, Berlin, corr. 2nd print edn.

Maki, D. (2009): "Tests for a Unit Root Using Three-Regime TAR Models: Power Comparison and Some Applications," *Econometric Reviews*, 28(4), 335–363.

Million, N. (2004): "Central Bank's interventions and the Fisher hypothesis: a threshold cointegration investigation," *Economic Modelling*, 21(6), 1051–1064.

Nelson, and Plosser (1982): "Trends and Random Walks in Macroeconomic Time Series: Some Evidence and Implications.," *Journal of Monetary Economics.*, 10(2), 139–162.

Perron, P. (1989): "The Great Crash, the Oil Price Shock, and the Unit Root Hypothesis," *Econometrica*, 57(6), 1361–1401.

Petrucelli, J., and N. Davies (1986): "A portmanteau test for self-exciting threshold autoregressive-type nonlinearity in time series," *Biometrika*, 73(3), 687–694.

Pfaff, B. (2008a): *Analysis of Integrated and Cointegrated Time Series with R*. Springer, New York, second edn., ISBN 0-387-27960-1.

——— (2008b): "VAR, SVAR and SVEC Models: Implementation Within R Package vars," *Journal of Statistical Software*, 27(4).

Phillips, P. (1986): "Understanding spurious regressions in econometrics," *Journal of Econometrics*, 33(3), 311–340.

Phillips, P., and S. Ouliaris (1990): "Asymptotic Properties of Residual Based Tests for Cointegration," *Econometrica*, 58(1), 165–193.

Pippenger, M. K., and G. E. Goering (2000): "Additional Results on the Power of Unit Root and Cointegration Tests under Threshold Processes," *Applied Economics Letters*, 7(10), 641–44.

Samuelson, P. (1965): "Proof That Properly Anticipated Prices Fluctuate Randomly," *Industrial Management Review*, 6, 41–49.

Schmidberger, M., M. Morgan, D. Eddelbuettel, H. Yu, L. Tierney, and U. Mansmann (2009): "State of the Art in Parallel Computing with R," *Journal of Statistical Software*, 31(1), 1–27.

Seo, M. (2006): "Bootstrap testing for the null of no cointegration in a threshold vector error correction model," *Journal of Econometrics*, 127(1), 129–150.

——— (2008): "Unit Root Test In A Threshold Autoregression: Asymptotic Theory And Residual-Based Block Bootstrap," *Econometric Theory*, 24(06), 1699–1716.

——— (2009): "Estimation of non linear error-correction models," Discussion paper, London School of Economics, Consulted in July 2009.

Seo, M., and O. Linton (2007): "A smoothed least squares estimator for threshold regression models," *Journal of Econometrics*, 141(2), 704–735.

Sims, C. A. (1980): "Macroeconomics and Reality," *Econometrica*, 48(1), 1–48.

Stock, J. H. (1987): "Asymptotic Properties of Least Squares Estimators of Cointegrating Vectors," *Econometrica*, 55:5, 1035–1056.

Taylor, A. M. (2001): "Potential Pitfalls for the Purchasing-Power-Parity Puzzle? Sampling and Specification Biases in Mean-Reversion Tests of the Law of One Price," *Econometrica*, 69(2), 473–498.

Tong, H. (1990): *Non-Linear Time Series. A Dynamical System Approach*. Oxford Science Publications.

Tsay, R. S. (1989): "Testing and Modeling Threshold Autoregressive Processes," *Journal of the American Statistical Association*, 84(405), 231–240.

Ward, R. (1982): "Asymmetry in Retail, Wholesale and Shipping Point Pricing for fresh Vegetables," *American Journal of Agricultural Economics*, 64(2), 205–212.

Zivot, E., and D. W. Andrews (1992): "Further Evidence on the Great Crash, the Oil-Price Shock, and the Unit-Root Hypothesis," *Journal of Business & Economic Statistics*, 10(3), 251–270.