An incarnation of the Little's Law for the phenomena of queue when queue exists.

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Abstract.

In queue phenomena, it is interesting to know the waiting time or the mean number of customers in queue knowing that queue exists. The results derived in this paper can be used in any basic markovian model.

1 Introduction.

In many books about queueing theory, the waiting time in a queue or the mean number of customers in queue knowing that queue exists is omitted or not completely developed [GROSS2008], [SIXTO2004], [MEDHI2003].

In this paper a complete framework for several basic markovian models is developed**.** Following the definition of the nomenclature used, several definitions and auxiliary theorems will be developed in the Queueing Metrics section to be used in the main section of this paper.

After this, a complete compendium of formulas for the mean number of customers in queue when queue exists (denoted by L_{qq}) and for the mean number of time in queue when queue exists (denoted by W_{qq}) will be obtained and those results will be checked using the R software package *queueing*.

Finally, the acknowledgments and references sections close this paper.

2 Queueing Metrics.

Using the nomenclature given in [SIXTO2004], and reviewing the following generic and basic queueing model:

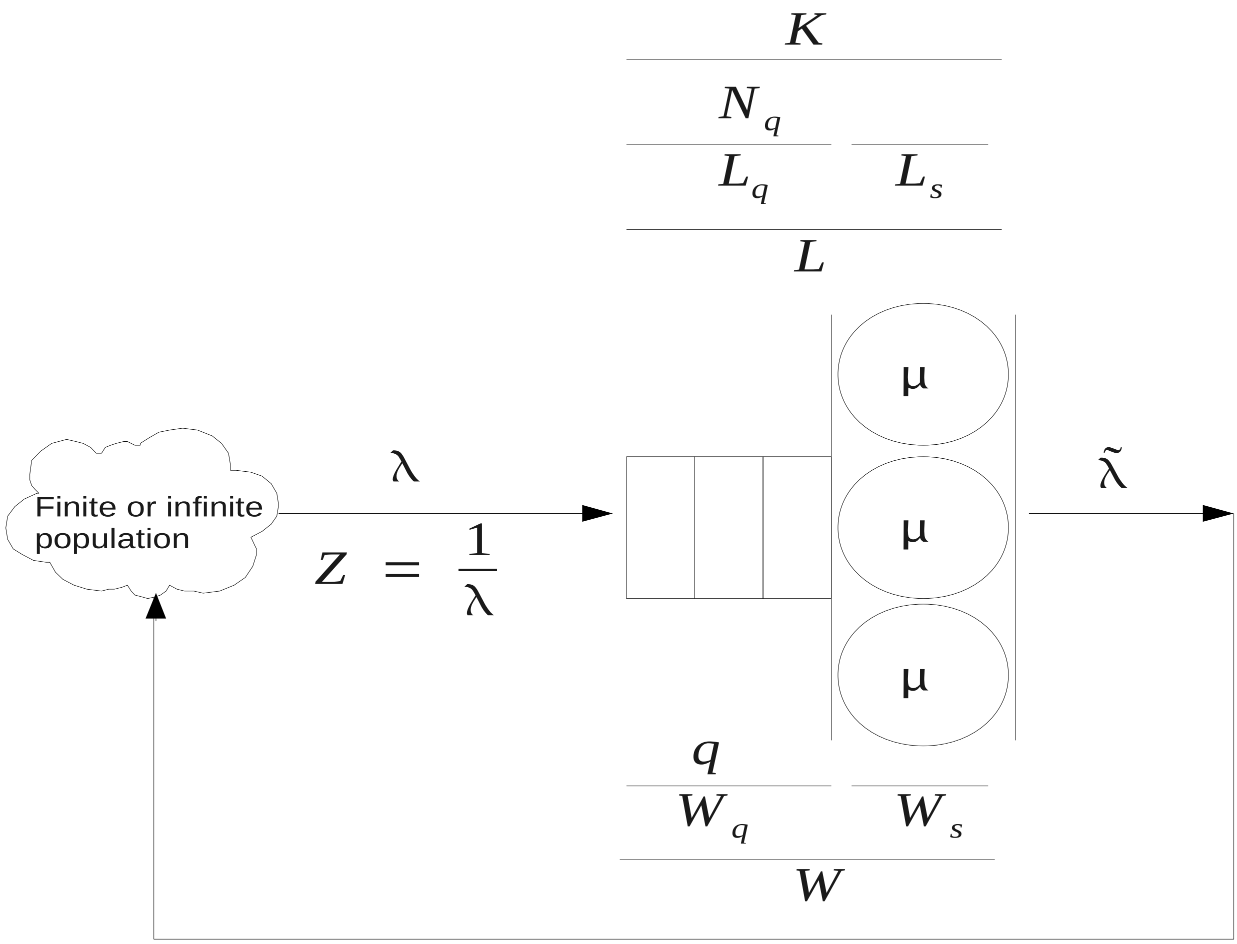

Illustration 1

let

λ = arrival rate of customers to the system,
Z = mean time between successive arrivals of customers to the system,
μ = service rate,
c = number of channels of the server (for example, in illustration 1, the server has three identical channels)
$\tilde{\lambda}$ = throughput of the system or rate of completion of the system.
K = capacity of the system, consisting of maximum number of available positions in queue and in the server (In illustration 1, the capacity of the system is six positions: three positions in queue and three positions in the server).
m = size of population. It takes a finite value and it is represented if the calling population is finite.
N_q = random variable representing the number of customers in queue.
L_q = mean number of customers in queue.
L_s = mean number of customers in service.

L = Mean number of customers in the system (in queue or receiving service).
q = Random variable representing time spent in queue.
W_q = Mean time in queue.
W_s = Mean time in service.
W = Mean time in system (including both mean time in queue and in service).

Observe that in each line for the N_q, L_q, q and W_q metrics, the random variables are situated at top of the line and the mean of that random variables are situated under the line.

In addition to the metrics represented in the illustration 1, the following metrics also will be used in the paper:

ρ = server use or utilization,
π_i = probability of *i* customers in the system,
q_i = probability that an input customer (that is, that effectively enters in the system because the system has not reached their maximum capacity) finds *i* customers.
L_{qq} = mean number of customers in queue when queue exists.
W_{qq} = mean waiting time in queue when queue exists.

With these nomenclature,

Definition 1[1]: The probability that the number of customers in queue are greater than zero, can be defined as

$$P[N_q > 0] \;=\; \pi_{c+1} + \pi_{c+2} + \dots \;=\; \sum_{i=1}^{\infty} \pi_{c+i}$$

This follows because the number of people waiting in queue will be greater than zero when the system has *c+1*, or *c+2*, ... customers in the system, or equivalently, when there are more customers than channels.

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Definition 2: The probability that time spent in queue is greater than zero, can be defined as

$$P[q > 0] = \; q_c + q_{c+1} + \dots \;=\; \sum_{i=0}^{\infty} q_{c+i}$$

This follows because time spent in queue will be greater than zero when an arrival see the *c*, or *c+1*, or ... customers in the system, or equivalently, when an arrival see all channels busy.

■

Theorem 1. As show in [SIXTO2004], W_{qq} and L_{qq} are defined:

$$W_{qq} \;=\; E[q \mid q > 0] \;=\; \frac{W_q}{P[q > 0]} \quad \text{, and} \quad L_{qq} \;=\; E[N_q \mid N_q > 0] \;=\; \frac{L_q}{P[N_q > 0]} \quad .$$

1 Thanks to See Chuen-Teck who inspired me giving me a complete formula for Lqq in the *M/M/c* model.

Proof:

$$W_q = E[q] = E[q \mid q = 0] \cdot P[q = 0] + E[q \mid q > 0] \cdot P[q > 0] \quad .$$

Because $E[q \mid q = 0] = 0$, and naming $W_{qq} = E[q \mid q > 0]$, it follows

$$W_q = W_{qq} \cdot P[q > 0] \Leftrightarrow W_{qq} = \frac{W_q}{P[q > 0]} \quad .$$

In a similar way,

$$L_q = E[N_q] = E[N_q \mid N_q = 0] \cdot P[N_q = 0] + E[N_q \mid N_q > 0] \cdot P[N_q > 0] \quad .$$

Because $E[N_q \mid N_q = 0] = 0$, and naming $L_{qq} = E[N_q \mid N_q > 0]$, it follows,

$$L_q = L_{qq} \cdot P[N_q > 0] \Leftrightarrow L_{qq} = \frac{L_q}{P[N_q > 0]} \quad .$$

■

Theorem 2. For the model *M/M/c/K/m* (m > K > c),

$$q_n = \frac{(m-n)\pi_n}{(m-L) - \pi_K(m-K)}$$

Proof:

Using the definition of q_n given in [MEDHI2003], pag. 80, eq. 3.3.7,

$$q_n = \frac{P[an\,arrival\,ocurring \mid system\,with\,n\,customers] \cdot \pi_n}{\sum_{i=0}^{K-1} P[an\,arrival\,ocurring \mid system\,with\,i\,customers] \cdot \pi_i}$$

and taking note that, because the arrivals are distributed as Poisson, denoting *h* as an infinitesimal period of time and *o(h)* as a function of *h* that converges to zero more rapidly than *h*,

$\frac{o(h)}{h} \to 0 \;\; as \;\; h \to 0$, then from

$\lim_{h \to 0} P[an\,arrival\,ocurring \mid system\,with\,i\,customers] = \{(m-i) \cdot \lambda \cdot h + o(h)\}$, it follows

$$q_n = \lim_{h \to 0} \frac{\{(m-n)\lambda h + o(h)\}\pi_n}{\sum_{i=0}^{K-1} \{(m-i)\lambda h + o(h)\}\pi_i} = \frac{(m-n)\lambda\pi_n}{\lambda \sum_{i=0}^{K-1} m\pi_i - \lambda \sum_{i=0}^{K-1} i\pi_i} =$$
$$\frac{(m-n)\lambda\pi_n}{\lambda m(1-\pi_K) - \lambda(L - K\pi_K)} = = \frac{(m-n)\lambda\pi_n}{\lambda[m(1-\pi_K) - (L - K\pi_K)]} =$$
$$\frac{(m-n)\pi_n}{m(1-\pi_K) - (L - K\pi_K)} = \frac{(m-n)\pi_n}{(m-L) - \pi_K(m-K)}$$

■

Theorem 3. For the model *M/M/c/K/m* (m > K > c), $\tilde{\lambda} = \lambda\,[(m-L) + \pi_K(m-K)]$.

Proof:

To obtain it, observe that in all finite markovian models, the throughput is lambda (arrival rate) times the denominator of the q_n.

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Corollary 1: The Little's Law for the model *M/M/c/K/m* has an interesting interpretation.

$$\frac{\tilde{\lambda}}{\lambda} = \tilde{\lambda} Z = [(m-L) + \pi_K(m-K)] \Leftrightarrow$$
$$\tilde{\lambda} Z + L + K\pi_K = m + \pi_K m = m(1+\pi_K) \Leftrightarrow$$
$$m = \frac{\tilde{\lambda} Z}{(1+\pi_K)} + \frac{L}{(1+\pi_K)} + K\frac{\pi_K}{(1+\pi_K)}$$

The first term represents the number of customers in the source, the second represents the number of customers in the system (in queue or receiving service), and the last term represents the rejected customers because the lack of space. Because, $0 \leqslant \pi_K \leqslant 1$, each one of the terms are reduced by the factor $(1+\pi_K)$.

When $\pi_K \simeq 0$, $m \simeq \tilde{\lambda} Z + L$, and when $\pi_K \simeq 1$, $m \simeq \frac{\tilde{\lambda} Z + L + K}{2}$.

Observe that in the case of *m = K*, the model is *M/M/c/K/K*, and consequently,

$$\frac{\tilde{\lambda}}{\lambda} = \tilde{\lambda} Z = [(K-L)] \Leftrightarrow$$
$$K = \tilde{\lambda} Z + L$$

that is, *K* is the sum of the mean number customers in the source and the mean number of customers in the system, according with the formula of the page 417 of [SIXTO2004].

Theorem 4: For model *M/M/c/K*, naming $u = \frac{\lambda}{\mu}$ and $\omega = \frac{u}{c} = \frac{\lambda}{\mu \cdot c}$, then

$$\frac{\pi_{c+j}}{\pi_c} = \omega^j \quad , \quad 0 \leqslant j \leqslant (K-c) \quad .$$

Proof:

For j = 0, the proof is trivial.
For j > 0, and knowing that the stationary probabilities when *n* > *c* are defined as:

$\pi_n = \frac{u^n}{c!c^{n-c}}\pi_0$, it follows

$$\frac{\pi_{c+j}}{\pi_c} = \frac{\frac{u^{c+j}}{c!c^j}\pi_0}{\frac{u^c}{c!}\pi_0} = \frac{u^j}{c^j} = \omega^j \quad .$$

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3 Little's Law incarnation for basic markovian models where queue exists .

Main Theorem: For models

- *M/M/1*,
- *M/M/c*,
- *M/M/1/K* (K > 1),
- *M/M/c/K* (K > c),
- *M/M/1/K/K* (K > 1),
- *M/M/c/K/K* (K > c) and
- *M/M/c/K/m* (m > K > c),

$$\frac{P[N_q > 0]}{P[q > 0]} = \rho$$

Proof:

Using Definition 1 and Definition 2,

$$\frac{P[N_q > 0]}{P[q > 0]} = \frac{\pi_{c+1} + \pi_{c+2} + \dots}{q_c + q_{c+1} + \dots}$$

In the case of a *M/M/1* model, $\pi_1 = q_1$ by the *PASTA* property, so

$$\frac{P[N_q > 0]}{P[q > 0]} = \frac{\pi_2 + \pi_3 + \dots}{\pi_1 + \pi_2 + \dots} = \frac{1 - \pi_0 - \pi_1}{1 - \pi_0} = 1 - \frac{\pi_1}{\rho} = 1 - \frac{(1-\rho)\rho}{\rho} = \rho \quad .$$

Observe that in this model,

$\frac{\pi_{i+1}}{\pi_i} = \frac{(1-\rho)\rho^{i+1}}{(1-\rho)\rho^i} = \rho$, for $i \geq 0$. This result can also be seen in [MEDHI2003], pag. 66.

□

In case of a *M/M/c* model, the *PASTA* property also holds, so

$$\frac{P[N_q > 0]}{P[q > 0]} = \frac{\pi_{c+1} + \pi_{c+2} + \ldots}{\pi_c + \pi_{c+1} + \ldots} = \frac{P[N \geqslant c+1]}{P[N \geqslant c]} = \frac{\frac{\pi_{c+1}}{(1-\rho)}}{\frac{\pi_c}{(1-\rho)}} = \frac{\pi_{c+1}}{\pi_c} = \frac{\frac{r^{c+1}\pi_0}{c!\cdot c}}{\frac{r^c \pi_0}{c!}} = \frac{r}{c} = \rho$$

This result, obtained here as $\rho\pi_{c+i} = \pi_{c+i+1}$, $i \geq 0$, has also been noted in [MEDHI2003] pag. 85.

□

For the model *M/M/c/K/K*,

$\frac{P[N_q > 0]}{P[q > 0]} = \frac{\pi_{c+1} + \pi_{c+2} + \ldots + \pi_K}{q_c + q_{c+1} + \ldots + q_{K-1}}$. If $\frac{P[N_q > 0]}{P[q > 0]} = \rho$, then the following expression has to be true:

$$\frac{\pi_{c+1} + \pi_{c+2} + \ldots + \pi_K}{q_c + q_{c+1} + \ldots + q_{K-1}} = \rho \quad \Leftrightarrow \quad \pi_{c+1} + \pi_{c+2} + \ldots + \pi_K = \rho \cdot (q_c + q_{c+1} + \ldots + q_{K-1}) \quad .$$

That is,

$$\pi_{c+1} + \pi_{c+2} + \ldots + \pi_K = \rho \cdot (q_c + q_{c+1} + \ldots + q_{K-1}) = \rho q_c + \rho q_{c+1} + \ldots + \rho q_{K-1} \quad .$$

The goal is to show that $\pi_{c+1} = \rho q_c$, $\pi_{c+2} = \rho q_{c+1}$, ..., $\pi_K = \rho q_{K-1}$ (1).

In this model, $q_i = \frac{(K-i)\cdot\pi_i}{(K-L)}$, and for $i = c$, ..., K, the probabilities are defined as:

$\pi_i = \frac{i!}{c!\cdot c^{i-c}}\binom{K}{i}\left(\frac{\lambda}{\mu}\right)^i \pi_0$ and finally, $\rho = \frac{\lambda}{c\cdot\mu}\cdot(K-L)$. Then,

$$\rho q_i = \rho\cdot\frac{(K-i)\cdot\pi_i}{(K-L)} = \frac{\lambda}{c\cdot\mu}\cdot(K-L)\cdot\frac{(K-i)}{(K-L)}\cdot\frac{i!}{c!\cdot c^{i-c}}\binom{K}{i}\left(\frac{\lambda}{\mu}\right)^i \pi_0 =$$

$$= \frac{\lambda}{c\cdot\mu}\cdot(K-i)\cdot\frac{i!}{c!\cdot c^{i-c}}\binom{K}{i}\left(\frac{\lambda}{\mu}\right)^i \pi_0 = \frac{(K-i)}{c}\cdot\frac{i!}{c!\cdot c^{i-c}}\binom{K}{i}\left(\frac{\lambda}{\mu}\right)^{i+1} \pi_0 =$$

$$= \frac{(K-i)}{c}\cdot\frac{i!}{c!\cdot c^{i-c}}\left(\frac{\lambda}{\mu}\right)^{i+1}\cdot\frac{K!}{(K-i)!\cdot i!}\cdot\pi_0 = \frac{(K-i)}{c}\cdot\frac{i!}{c!\cdot c^{i-c}}\left(\frac{\lambda}{\mu}\right)^{i+1}\cdot\frac{K!}{(K-i)\cdot(K-(i+1))!\cdot i!}\cdot\pi_0 =$$

$$= \frac{(K-i)}{c} \cdot \frac{i!}{c! \cdot c^{i-c}} \left(\frac{\lambda}{\mu}\right)^{i+1} \cdot \frac{K!}{(K-i) \cdot (K-(i+1))! \cdot i!} \cdot \pi_0 = \frac{(i+1)!}{c! \cdot c^{i+1-c}} \left(\frac{\lambda}{\mu}\right)^{i+1} \cdot \binom{K}{i+1} \cdot \pi_0 = \pi_{i+1}$$

The model *M/M/1/K/K* is a special case of the model *M/M/c/K/K* by setting $c = 1$.

□

For the model *M/M/c/K/m*, the goal is to show that (1) also holds.

$$\rho q_i = \frac{\lambda[(m-L) - \pi_K(m-K)]}{c\mu} \frac{(m-i)\pi_i}{(m-L) - \pi_K(m-K)} = \frac{\lambda}{c\mu}(m-i)\pi_i =$$

$$= \frac{\lambda}{c\mu}(m-i)\frac{i!}{c!c^{i-c}}\binom{m}{i}\left(\frac{\lambda}{\mu}\right)^i \pi_0 = \left(\frac{\lambda}{\mu}\right)^{i+1} \frac{i!}{c!c^{i+1-c}}(m-i)\frac{m!}{i!(m-i)!}\pi_0 =$$

$$= \left(\frac{\lambda}{\mu}\right)^{i+1} \frac{i!}{c!c^{i+1-c}}(m-i)\frac{m!}{i!(m-i)[m-(i+1)]!}\pi_0 =$$

$$= \left(\frac{\lambda}{\mu}\right)^{i+1} \frac{i!}{c!c^{i+1-c}} \frac{m!}{i![m-(i+1)]!} \frac{(i+1)}{(i+1)}\pi_0 =$$

$$= \left(\frac{\lambda}{\mu}\right)^{i+1} \frac{(i+1)!}{c!c^{i+1-c}} \frac{m!}{(i+1)![m-(i+1)]!}\pi_0 = \left(\frac{\lambda}{\mu}\right)^{i+1} \frac{(i+1)!}{c!c^{i+1-c}} \binom{m}{i+1}\pi_0 = \pi_{i+1}$$

□

In the case of the *M/M/c/K* model, recalling that

$q_i = \dfrac{\pi_i}{1-\pi_K}$, $0 \leqslant i \leqslant K-1$, it follows

$$\frac{P[N_q > 0]}{P[q > 0]} = \frac{\pi_{c+1} + \pi_{c+2} + \dots + \pi_K}{q_c + q_{c+1} + \dots + q_{K-1}} = \frac{\pi_{c+1} + \pi_{c+2} + \dots + \pi_K}{\frac{\pi_c}{1-\pi_K} + \frac{\pi_{c+1}}{1-\pi_K} + \dots + \frac{\pi_{K-1}}{1-\pi_K}} =$$

$$= (1-\pi_K)\frac{\pi_{c+1} + \pi_{c+2} + \dots + \pi_K}{\pi_c + \pi_{c+1} + \dots + \pi_{K-1}} \ .$$

Now, naming $u = \dfrac{\lambda}{\mu}$ and $\omega = \dfrac{u}{c} = \dfrac{\lambda}{\mu \cdot c}$, and using Theorem 4, we have

$$(1-\pi_K)\frac{\pi_{c+1} + \pi_{c+2} + \dots + \pi_K}{\pi_c + \pi_{c+1} + \dots + \pi_{K-1}} = (1-\pi_K)\frac{\frac{\pi_{c+1}}{\pi_c} + \frac{\pi_{c+2}}{\pi_c} + \dots + \frac{\pi_K}{\pi_c}}{\frac{\pi_c}{\pi_c} + \frac{\pi_{c+1}}{\pi_c} + \dots + \frac{\pi_{K-1}}{\pi_c}} =$$

$$= (1-\pi_K)\frac{\omega + \omega^2 + \dots + \omega^K}{1 + \omega^2 + \dots + \omega^{K-1}}$$, where $\pi_c > 0$ has been assumed.

The formula for the sum of the *n* first terms of a geometric series is:

$S_n = \dfrac{a_1}{1-t} \cdot (1-t^n)$, where a_1 is the 1st term of the series, and *t* is the rate of growth of the series. Identifying terms,

$$(1-\pi_K)\frac{\omega+\omega^2+\ldots+\omega^K}{1+\omega^2+\ldots+\omega^{K-1}} = (1-\pi_K)\frac{\frac{\omega}{(1-\omega)}\cdot(1-\omega^K)}{\frac{1}{(1-\omega)}\cdot(1-\omega^K)} = (1-\pi_K)\cdot\omega = \rho \quad .$$

Recall that in one of the previous steps, $\pi_c > 0$ has been assumed.

It can be show that (1) also holds for this model, so the previous requirement is not really important.

$$\rho q_i = (1-\pi_K)\cdot\frac{u}{c}\frac{\pi_i}{1-\pi_K} = \frac{u}{c}\pi_i = \frac{u}{c}\cdot\frac{u^i}{c!\cdot c^{i-c}}\cdot\pi_0 = \frac{u^{i+1}}{c!\cdot c^{i+1-c}}\cdot\pi_0 = \pi_{i+1}$$

The *M/M/1/K* model is a special case of the model *M/M/c/K* by setting c = 1.

■

Corollary[2]. For the models of the Main Theorem , $L_{qq} = c\cdot\mu\cdot W_{qq}$.

Proof.

Using the formula of Theorem 1, then, $W_q = W_{qq}\cdot P[q>0]$ and $L_q = L_{qq}\cdot P[N_q>0]$.

By virtue of Little's Law, $\tilde{\lambda} = \frac{L_q}{W_q} = \frac{L_{qq}}{W_{qq}}\frac{P[N_q>0]}{P[q>0]}$, by Main Theorem, $\tilde{\lambda} = \frac{L_{qq}}{W_{qq}}\rho$.

Finally, by definition, $\rho = \frac{\tilde{\lambda}}{c\cdot\mu}$.

Then,

$$L_{qq} = c\cdot\mu\cdot W_{qq} \quad .$$

This formula is the incarnation of the Little's Law for the phenomena of queue when queue exists.

Corollary. For the models of the Main Theorem,

$$\rho q_i = \pi_{i+1} \quad , \quad \forall i \geq c$$

Proof.

It is derived by each model in the proof of the Main Theorem.

2 Thanks to See Chuen-Teck (see_chuenteck@yahoo.com.sg) for give me an intuitive justification of why this should be.

4 L_{qq} and W_{qq} formulas for Basic Markovian Models

In the proof of the Main Theorem, some interesting results has been obtained and in consequence, some closed expressions can be given here, in addition to some formulas already derived in the bibliography. The formulas marked as (*) are derived in [SIXTO2004].

Model M/M/1:

$$L_{qq} = \frac{L_q}{P[N_q > 0]} = \frac{\rho^2}{(1-\rho)\cdot P[N \geqslant 2]} = \frac{\rho^2}{(1-\rho)\cdot\rho^2} = \frac{1}{1-\rho} \quad (*).$$

$$W_{qq} = \frac{W_q}{P[q>0]} = \frac{\rho}{\mu\cdot(1-\rho)\cdot(1-\pi_0)} = \frac{\rho}{\mu\cdot(1-\rho)\cdot\rho} = W \quad (*),$$

Observe that the time a customer has to wait in queue when queue exists is the same as the mean time in the system. Because the mean time in the system is the sum of the mean time in queue and the mean time in service, that means that the customer that has to wait, has to wait an extra mean service time.

Writing the formula for L_q, we have:

$L_q = \frac{\rho^2}{1-\rho} = \rho^2\cdot L_{qq} \Leftrightarrow \frac{L_q}{\rho^2} = L_{qq}$. Because $0 \le \rho < 1$, that means that the mean number of customers seen by a customer that has to wait because queue exists , is $\frac{1}{\rho^2}$ times greater than the mean number of customers in queue (L_q).

Model M/M/c

$$L_{qq} = \frac{L_q}{P[N_q > 0]} = \frac{C(c,r)\cdot\rho}{(1-\rho)\cdot P[N \ge c+1]} = \frac{C(c,r)\cdot\rho}{(1-\rho)\cdot\frac{\pi_{c+1}}{(1-\rho)}} = \frac{C(c,r)\cdot\rho}{\pi_{c+1}} = \frac{1}{1-\rho}$$

$$W_{qq} = \frac{W_q}{P[q>0]} = \frac{C(c,r)}{c\cdot\mu\cdot(1-\rho)\cdot P[N \ge c]} = \frac{C(c,r)}{c\cdot\mu\cdot(1-\rho)\cdot C(c,r)} = \frac{1}{c\cdot\mu\cdot(1-\rho)} \quad (*).$$

Observe that the formula for L_{qq} in the *M/M/c* model adopts the same functional expression as the formula for the *M/M/1* model.

Doing some algebraic manipulations in L_{qq} and recalling the formula for L_q:

$$L_{qq} = \frac{1}{1-\rho} = \frac{L_q}{C(c,r)\cdot\rho} ,$$

That means that $\frac{1}{C(c,r)\cdot\rho}$ is the proportion of additional users that a customer is going to meet in queue because queue exists.

Note that as $W_{qq} = \dfrac{W_q}{C(c,r)}$, it results that $\dfrac{1}{C(c,r)}$ is the proportion of time that W_{qq} is greater than W_q.

Model M/M/1/K:

$$L_{qq} = \frac{L_q}{P[N_q > 0]} = \frac{L_q}{\sum_{i=2}^{K} \pi_i} = \frac{L_q}{1 - \pi_0 - \pi_1}$$

$$W_{qq} = \frac{W_q}{P[q > 0]} = \frac{W_q}{\sum_{i=1}^{K-1} q_i} = \frac{W_q}{(1 - q_0)} = \frac{W_q}{(1 - \dfrac{\pi_0}{1 - \pi_K})} = \frac{W_q \cdot (1 - \pi_K)}{(1 - \pi_0 - \pi_K)}$$

Model M/M/c/K:

$$L_{qq} = \frac{L_q}{P[N_q > 0]} = \frac{L_q}{\sum_{i=c+1}^{K} \pi_i} = \frac{L_q}{1 - \sum_{i=0}^{c} \pi_i}$$

$$W_{qq} = \frac{W_q}{P[q > 0]} = \frac{W_q}{\sum_{i=0}^{K-1} q_{c+i}} = \frac{W_q}{1 - \sum_{i=0}^{c-1} q_i}$$

Model M/M/1/K/K:

$$L_{qq} = \frac{L_q}{P[N_q > 0]} = \frac{L_q}{\sum_{i=2}^{K} \pi_i} = \frac{L_q}{1 - \pi_0 - \pi_1}$$

$$W_{qq} = \frac{W_q}{P[q > 0]} = \frac{W_q}{\sum_{i=1}^{K-1} q_i} = \frac{W_q}{(1 - q_0)} = \frac{W_q}{(1 - \dfrac{K\pi_0}{K - L})} = \frac{W_q \cdot (K - L)}{K - L - K\pi_0}$$

Models *M/M/c/K/K* and *M/M/c/K/m*:

$$L_{qq} = \frac{L_q}{P[N_q > 0]} = \frac{L_q}{\sum_{i=c+1}^{K} \pi_i} = \frac{L_q}{1 - \sum_{i=0}^{c} \pi_i}$$

$$W_{qq} = \frac{W_q}{P[q > 0]} = \frac{W_q}{\sum_{i=0}^{K-1} q_{c+i}} = \frac{W_q}{1 - \sum_{i=0}^{c-1} q_i}$$

5 Testing the results obtained using Queueing software.

The results of this paper has been tested using *queueing*, an open source R package, freely available in CRAN.

To check the results obtained, the following examples can be run in the R console.

Each example has been divided in two parts: one written to check Main Theorem and other to check the incarnation of Little's Law.

The output of each example that confirms the results of the paper are highlighted in bold.

For a model *M/M/1*, an example with $\lambda = \frac{1}{4}$, $\mu = \frac{1}{3}$ and setting the model to calculate the first ten stationary probabilities (in addition to π_0) :

```
> i_mm1 <- NewInput.MM1(lambda=1/4, mu=1/3, n=10)
> o_mm1 <- QueueingModel(i_mm1)
> Pn(o_mm1)
 [1] 0.25000000 0.18750000 0.14062500 0.10546875 0.07910156 0.05932617
 [7] 0.04449463 0.03337097 0.02502823 0.01877117 0.01407838
> RO (o_mm1) * Qn(o_mm1)[1:10]
 [1] 0.18750000 0.14062500 0.10546875 0.07910156 0.05932617 0.04449463
 [7] 0.03337097 0.02502823 0.01877117 0.01407838

> Wqq <- Wq(o_mm1)/(1 - Pn(o_mm1)[1])
> Lqq <- Lq(o_mm1)/(1 - Pn(o_mm1)[1] - Pn(o_mm1)[2])
> Lqq
[1] 4
> Wqq * Inputs(o_mm1)$mu
[1] 4
```

For the model *M/M/c*, $\lambda = 5$, $\mu = 10$, $c = 2$, and setting the model to calculate the first ten stationary probabilities (in addition to π_0):

```
> i_mmc <- NewInput.MMC(lambda=5, mu=10, c=2, n=10, method=0)
> o_mmc <- QueueingModel(i_mmc)
> Pn(o_mmc)
 [1] 6.000000e-01 3.000000e-01 7.500000e-02 1.875000e-02 4.687500e-03
 [6] 1.171875e-03 2.929687e-04 7.324219e-05 1.831055e-05 4.577637e-06
[11] 1.144409e-06
> RO (o_mmc) * Qn(o_mmc)[1:10]
 [1] 1.500000e-01 7.500000e-02 1.875000e-02 4.687500e-03 1.171875e-03
 [6] 2.929687e-04 7.324219e-05 1.831055e-05 4.577637e-06 1.144409e-06

> Wqq <- Wq(o_mmc)/(Pn(o_mmc)[Inputs(o_mmc)$c+1]/(1-RO(o_mmc)))
> Lqq <- Lq(o_mmc)/(Pn(o_mmc)[Inputs(o_mmc)$c+2]/(1-RO(o_mmc)))
> Lqq
```

[1] **1.333333**
> Wqq * Inputs(o_mmc)$c * Inputs(o_mmc)$mu
[1] **1.333333**

For the model *M/M/1/K*, an example with $\lambda = 5$, $\mu = 5.714$ and $K = 15$ is run,

> i_mm1k <- NewInput.MM1K(lambda=5, mu=5.714, k=15)
> o_mm1k <- QueueingModel(i_mm1k)
> Pn(o_mm1k)
 [1] 0.14169971 0.12399345 **0.10849969 0.09494198 0.08307838 0.07269722**
 [7] **0.06361325 0.05566438 0.04870876 0.04262230 0.03729638 0.03263596**
[13] **0.02855789 0.02498941 0.02186682 0.01913443**
> RO(o_mm1k) * Qn(o_mm1k)
 [1] 0.12399345 **0.10849969 0.09494198 0.08307838 0.07269722 0.06361325**
 [7] **0.05566438 0.04870876 0.04262230 0.03729638 0.03263596 0.02855789**
[13] **0.02498941 0.02186682 0.01913443**

> Wqq <- Wq(o_mm1k)/sum(Qn(o_mm1k)[2:(Inputs(o_mm1k)$k)])
> Lqq <- Lq(o_mm1k)/sum(Pn(o_mm1k)[3:(Inputs(o_mm1k)$k+1)])
> Lqq
[1] **5.448114**
> Wqq * Inputs(o_mm1k)$mu
[1] **5.448114**

For the model *M/M/c/K*, with parameters $\lambda = 8$, $\mu = 4$, $c = 5$, and $K = 12$:

> i_mmck <- NewInput.MMCK(lambda=8, mu=4, c=5, k=12)
> o_mmck <- QueueingModel(i_mmck)
> Pn(o_mmck)
 [1] 1.343336e-01 2.686672e-01 2.686672e-01 1.791115e-01 8.955574e-02
 [6] **3.582230e-02 1.432892e-02 5.731568e-03 2.292627e-03 9.170508e-04**
[11] **3.668203e-04 1.467281e-04 5.869125e-05**
> RO(o_mmck) * Qn(o_mmck)
 [1] 5.373345e-02 1.074669e-01 1.074669e-01 7.164459e-02 **3.582230e-02**
 [6] **1.432892e-02 5.731568e-03 2.292627e-03 9.170508e-04 3.668203e-04**
[11] **1.467281e-04 5.869125e-05**

> Wqq <- Wq(o_mmck)/(sum(Qn(o_mmck)[(Inputs(o_mmck)$c+1):(Inputs(o_mmck)$k)]))
> Lqq <- Lq(o_mmck)/(sum(Pn(o_mmck)[(Inputs(o_mmck)$c+2):(Inputs(o_mmck)$k+1)]))
> Lqq
[1] **1.655179**
> Wqq * Inputs(o_mmck)$c * Inputs(o_mmck)$mu
[1] **1.655179**

For the model *M/M/1/K/K*, with parameters $\lambda = 0.25$, $\mu = 4$ and $K = 2$:

> i_mm1kk <- NewInput.MM1KK(lambda=0.25, mu=4, k=2, method=0)
> o_mm1kk <- QueueingModel(i_mm1kk)

> Pn(o_mm1kk)
[1] 0.882758621 **0.110344828 0.006896552**
> RO(o_mm1kk) * Qn(o_mm1kk)
[1] **0.110344828 0.006896552**

> Wqq <- Wq(o_mm1kk)/(sum(Qn(o_mm1kk)[2:(Inputs(o_mm1kk)$k)]))
> Lqq <- Lq(o_mm1kk)/(sum(Pn(o_mm1kk)[3:(Inputs(o_mm1kk)$k+1)]))
> Lqq
[1] **1**
> Wqq * Inputs(o_mm1kk)$mu
[1] **1**

For the model *M/M/c/K/K*, with parameters $\lambda = 8$, $\mu = 2$, $c = 5$ and $K = 12$:

> i_mmckk <- NewInput.MMCKK(lambda=8, mu=2, c=5, k=12, method=0)
> ## Build the model
> o_mmckk <- QueueingModel(i_mmckk)
> Pn(o_mmckk)
 [1] 3.342312e-10 1.604310e-08 3.529482e-07 4.705976e-06 4.235378e-05
 [6] 2.710642e-04 **1.517960e-03 7.286206e-03 2.914482e-02 9.326343e-02**
[11] **2.238322e-01 3.581316e-01 2.865053e-01**
> RO(o_mmckk) * Qn(o_mmckk)
 [1] 3.208620e-09 1.411793e-07 2.823585e-06 3.388302e-05 2.710642e-04
 [6] 1**.517960e-03 7.286206e-03 2.914482e-02 9.326343e-02 2.238322e-01**
[11] **3.581316e-01 2.865053e-01**

> Wqq <- Wq(o_mmckk)/(sum(Qn(o_mmckk)[(Inputs(o_mmckk)$c+1):(Inputs(o_mmckk)$k)]))
> Lqq <- Lq(o_mmckk)/(sum(Pn(o_mmckk)[(Inputs(o_mmckk)$c+2):(Inputs(o_mmckk)$k+1)]))
> Lqq
[1] **5.751898**
> Wqq * Inputs(o_mmckk)$c * Inputs(o_mmckk)$mu
[1] **5.751898**

For the model *M/M/c/K/m*, $\lambda = 0.25$, $\mu = 4$, $c = 2$, $K = 4$ and $m = 8$:

> i_mmckm <- NewInput.MMCKM(lambda=0.25, mu=4, c=2, k=4, m=8, method=0)
> o_mmckm <- QueueingModel(i_mmckm)
> Pn(o_mmckm)
[1] 0.61233719 0.30616859 0.06697438 **0.01255770 0.00196214**
> RO(o_mmckm) * Qn(o_mmckm)
[1] 0.15308430 0.06697438 **0.01255770 0.00196214**

> Wqq <- Wq(o_mmckm)/(sum(Qn(o_mmckm)[(Inputs(o_mmckm)$c+1):(Inputs(o_mmckm)$k)]))
> Lqq <- Lq(o_mmckm)/(sum(Pn(o_mmckm)[(Inputs(o_mmckm)$c+2):(Inputs(o_mmckm)
$k+1)]))
> Lqq
[1] **1.135135**
> Wqq * Inputs(o_mmckm)$c * Inputs(o_mmckm)$mu
[1] **1.135135**

6 Conclusion

In this paper some useful relationships between probabilities when the system has queue has been developed. It is useful by itself because it permits to understand better the phenomena that occurs in that situation.

An useful incarnation of the Little's Law has been obtained for the models studied. Some work is in progress to test if this relationships also holds for more general models (*M/G/1*, *M/G/1/K*, etc).

A compilation of formulas for the L_{qq} and W_{qq} has been recompiled, and some interesting properties has been developed for the infinite population markovian models *M/M/1* and *M/M/c*.

And finally, the results has been checked using a software tool, which it permits to understand the phenomena under an easy framework.

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8 References

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