The galois package*

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1 Introduction

This `galois` package introduces two-dimensional notations for Galois connections.

2 Detailled explanations on Galois connections

If $(L, \leq)$ and $(M, \sqsubseteq)$ are posets, $\alpha \in L \mapsto M$, $\gamma \in M \mapsto L$ and $\forall x \in L, y \in M$: $\alpha(x) \sqsubseteq y \iff x \leq \gamma(y)$ then the pair $\langle \alpha, \gamma \rangle$ is a *Galois connection*, written
`\galois` `\galois{\alpha}{\gamma}`:

$$(L, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (M, \sqsubseteq)$$

In a Galois connection, α is onto if and only if γ is one-to-one if and only if $\alpha \circ \gamma = 1$ (where $\circ$ is the functional composition and 1 the identity), written
`\galoiS` `\galoiS{\alpha}{\gamma}`:

$$(L, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows\!\!\!\!\rightarrow}} (M, \sqsubseteq)$$

α is one-to-one if and only if γ is onto if and only if $\gamma \circ \alpha = 1$, written
`\Galois` `\Galois{\alpha}{\gamma}`:

$$(L, \leq) \underset{\alpha}{\overset{\gamma}{\leftarrow\!\!\!\!\rightarrow}} (M, \sqsubseteq)$$

`\GaloiS` For a bijection, we write `\GaloiS{\alpha}{\gamma}`:

$$(L, \leq) \underset{\alpha}{\overset{\gamma}{\leftarrow\!\!\!\!\rightarrow\!\!\!\!\rightarrow}} (M, \sqsubseteq)$$

The surjection on the quotient of M by the equivalence relation $x \equiv y$ defined by
`\galoiSr` $\gamma(x) = \gamma(y)$ is denoted `\galoiSr{\alpha}{\gamma}`:

$$(L, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows\!\!\!\!\equiv\!\!\!\!\rightarrow}} (M, \sqsubseteq)$$

The composition of Galois connections:

$$(L, \leq) \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} (M, \sqsubseteq) \quad \text{and} \quad (M, \sqsubseteq) \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} (N, \preceq)$$

`\comp` is a Galois connection (the composition $\circ$ of functions is `\comp`):

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$$(L,\leq)\ \underset{\alpha_2\circ\alpha_1}{\overset{\gamma_1\circ\gamma_2}{\longleftrightarrows}}\ (N,\preceq)$$

Galois connections $(L,\leq)\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ (M,\sqsubseteq)$ can be lifted from sets of properties to sets of monotone functions:

$$(L\overset{m}{\longmapsto}L,\leq)\ \underset{\lambda\varphi\cdot\alpha\circ\varphi\circ\gamma}{\overset{\lambda\phi\cdot\gamma\circ\phi\circ\alpha}{\longleftrightarrows}}\ (M\overset{m}{\longmapsto}M,\sqsubseteq)$$

where the ordering on functions is pointwise that is $\varphi\preceq\phi$ if and only if $\forall x:\varphi(x)\preceq\phi(x)$. Observe that the length of the arrows stretches automatically to the appropriate width.

3 Package options

`color` The `color` option is required for colored Galois connections is in

`\galois[red]{\alpha}{\gamma}`	$\underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}$,
`\Galois{\alpha}[blue]{\gamma}`	$\underset{\alpha}{\overset{\gamma}{\twoheadleftarrow\!\!\rightarrow}}$,
`\GaloiS[red]{\alpha}[blue]{\gamma}`	$\underset{\alpha}{\overset{\gamma}{\twoheadleftarrow\!\!\twoheadrightarrow}}$,
`\galoiSr[red]{\alpha}[blue]{\gamma}`	$\underset{\alpha}{\overset{\gamma}{\twoheadleftarrow\!=\!\!\twoheadrightarrow}}$, or
`\comp[red]`	$\circ$.

Without '`color`' option, these colors are ignored.

`\@GALOIScolor` `\@GALOIScolor` is `\color` with the `color` option and later defined as `\relax` in absence of `color` option.

```
1 \DeclareOption{color}{%
2   \def\@GALOIScolor{\color}}
3 \ProcessOptions
```

4 Style parameters

You can use Galois connections in any size (footnotes, transparencies, etc.) : tiny $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, scriptsize $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, footnotesize $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, small $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, normalsize $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, large $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, Large $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, LARGE $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, huge $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$, Huge $L\ \underset{\alpha}{\overset{\gamma}{\longleftrightarrows}}\ M$. Observe that in $\underset{\blacksquare}{\overset{\blacksquare}{\twoheadleftarrow\!\!\twoheadrightarrow}}$ the width of arrows and height of enclosing box are automatically adjusted according to the size of α and γ. You can adjust the following parameters:

`\GaloisStyle` `\GaloisStyle` : style of upper and lower tags (`\scripstyle` by default);

\GaloisArrowThickness

\GaloisArrowThickness : thickness of the arrow stems [1]; (0.1ex by default);

\GaloisArrowsSep

\GaloisArrowsSep : distance between the arrows (0.2ex by default);

\GaloisArrowTagSep

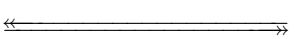
\GaloisArrowTagSep : distance between arrows and tags (0.5ex by default).

For example with:

```
\renewcommand{\GaloisArrowsSep}{1cm}
\renewcommand{\GaloisArrowTagSep}{0pt}
```

we get  while with:

```
\renewcommand{\GaloisArrowsSep}{0pt}
\renewcommand{\GaloisArrowTagSep}{5mm}
```

we get  and with

```
\renewcommand{\GaloisArrowsSep}{0pt}
\renewcommand{\GaloisArrowTagSep}{0pt}
```

we get .

5 Implementation

4 ⟨∗package⟩

Require color package for 'color' option else coloring is ignored.

```
5 \ifx\@GALOIScolor\undefined
6 \def\@GALOIScolor#1{\relax}%
7 \else
```

[1] stem is "tige" in french.

```
\RequirePackage{color}%
\fi
```

`\comp` `\@GALOIScomp` Define functional composition $f \circ g(x)$ is $f(g(x))$ (if not already defined e.g. as in `mathtime.sty`). `\comp[color]` will draw in color (black by default).

```
\@ifundefined{comp}{%
% Scan the optional color argument
\newcommand{\comp}{\@ifnextchar[{\@GALOIScomp}{\@GALOIScomp[black]}}%
% Defined the colored functional composition \@GALOIScomp[color]
\def\@GALOIScomp[#1]{\mathchoice
{\mathrel{\raisebox{0.2ex}{$\@GALOIScolor{#1}\scriptstyle\circ$}}}%
{\mathrel{\raisebox{0.2ex}{$\@GALOIScolor{#1}\scriptstyle\circ$}}}%
{\mathrel{\raisebox{0.1ex}{$\@GALOIScolor{#1}\scriptscriptstyle\circ$}}}%
{\mathrel{\raisebox{0.1ex}{$\@GALOIScolor{#1}\scriptscriptstyle\circ$}}}%
}{}%
```

Style commands:

`\GaloisStyle` Style of a and b in $\xleftarrow[a]{b}$, $\xleftarrow[a]{b}$, $\xleftarrow[a]{b}$ or $\xleftarrow[a]{b}$:

```
\newcommand{\GaloisStyle}{\scriptstyle}%
```

`\GaloisArrowThickness` Thickness of the arrow stems (0.1ex by default):

```
\newcommand{\GaloisArrowThickness}{0.1ex}%
```

`\GaloisArrowsSep` Distance between the lower and upper arrows (0.2ex by default):

```
\newcommand{\GaloisArrowsSep}{0.2ex}%
```

`\GaloisArrowTagSep` Distance between the lower arrow and the top of a and the top-arrow and the bottom of b (0.5ex by default)

```
\newcommand{\GaloisArrowTagSep}{0.5ex}%
```

`\@GALOISalphadepth` `\@GALOISalphaheight` `\@GALOISgammadepth` `\@GALOISwidth` `\@GALOISheight` `\@GALOISdepth` `\@GALOIStotalheight` `\@GALOISGap` `\@GALOISalphaarrowwidth` `\@GALOISalphaarrowhalfheight` `\@GALOISgammaarrowwidth` `\@GALOISgammaarrowhalfheight` Auxiliary lengths:

```
{}
\newlength{\@GALOISalphadepth}%
\newlength{\@GALOISalphaheight}%
\newlength{\@GALOISgammadepth}%
\newlength{\@GALOISwidth}%
\newlength{\@GALOISheight}%
\newlength{\@GALOISdepth}%
\newlength{\@GALOIStotalheight}%
\newlength{\@GALOISGap}%
\newlength{\@GALOISalphaarrowwidth}%
\newlength{\@GALOISalphaarrowhalfheight}%
\newlength{\@GALOISgammaarrowwidth}%
\newlength{\@GALOISgammaarrowhalfheight}%
```

`\Galois@put` `\Galois@put(x,y-d){text}` puts text at coordinates $(x, y - d)$, in a box of size `0pt` × `0pt`:

```
\def\Galois@put(#1,#2-#3)#4{\rlap{\smash{\hskip#1\setlength{\@tempdimc}{#2}%
\addtolength{\@tempdimc}{-#3}\raisebox{\@tempdimc}{#4}}}}%
```

`\@GALOISrightarrowfill` `\@GALOISrightarrowfill{\rightarrow}`, see TEXbook p. 357.

```
\def\@GALOISrightarrowfill#1{$\m@th \smash- \mkern-7mu%
 \cleaders\hbox{$\mkern-2mu \smash- \mkern-2mu$}\hfill%
 \mkern-7mu \mathord{#1}$}%
```

\@GALOISleftarrowfill `\@GALOISleftarrowfill{\leftarrow}`, see TEXbook p. 357.

```
\def\@GALOISleftarrowfill#1{$\m@th \mathord{#1} \mkern-7mu%
  \cleaders\hbox{$\mkern-2mu \smash- \mkern-2mu$}\hfill%
  \mkern-7mu \smash-$}%
```

Stacking a, the arrows and g in $\xleftarrow[a]{g}$:

\@GALOIS `\@GALOIS{-->}{<--}{a}{g}` constructs $\xleftarrow[a]{g}$. `\@GALOIS{-->}{<--}[colora]{a}{g}`,
\@GALOISca `\@GALOIS{-->}{<--}{a}[colorg]{g}` and `\@GALOIS{-->}{<--}[colora]{a}[colorg]{g}`
\@GALOISca add colors `colora` for the a-arrow and `colorg` for the$g$ arrow.

```
%    First, scan the alpha color optional argument (black
%    otherwise)
\def\@GALOIS#1#2{\@ifnextchar[{\@GALOISca{#1}{#2}}{\@GALOISca{#1}{#2}[black]}}%
%    Second scan the gamma color optional argument (black
%    otherwise)
\def\@GALOISca#1#2[#3]#4{\@ifnextchar[{\@GALOIScacg{#1}{#2}[#3]{#4}}%
                                      {\@GALOIScacg{#1}{#2}[#3]{#4}[black]}}%
%    Finally \@GALOIScacg{-->}{<--}[colora]{a}[colorg]{g} stacks $a$,
%    the arrows and $g$ in $\galois{a}{g}$, using colors with the
%    'color' option.
\def\@GALOIScacg#1#2[#3]#4[#5]#6{%
\ensuremath{\mathrel{%
\def\@GALOISalphatag{\ $\@GALOIScolor{#3}\GaloisStyle#4$\ }%
\def\@GALOISgammatag{\ $\@GALOIScolor{#5}\GaloisStyle#6$\ }%
% compute width of alpha/lower and gamma/upper arrows
\settowidth{\@GALOISalphaarrowwidth}{$\mathord{#1}$}%
\settowidth{\@GALOISgammaarrowwidth}{$\mathord{#2}$}%
% compute width of the picture \@GALOISwidth
\ifdim\@GALOISalphaarrowwidth>\@GALOISgammaarrowwidth%
\settowidth{\@tempdima}{\hbox{\hspace*{\@GALOISalphaarrowwidth}\@GALOISalphatag}}%
\settowidth{\@tempdimb}{\hbox{\hspace*{\@GALOISalphaarrowwidth}\@GALOISgammatag}}%
\else%
\settowidth{\@tempdima}{\hbox{\hspace*{\@GALOISgammaarrowwidth}\@GALOISalphatag}}%
\settowidth{\@tempdimb}{\hbox{\hspace*{\@GALOISgammaarrowwidth}\@GALOISgammatag}}%
\fi%
\ifdim\@tempdima>\@tempdimb%
\setlength{\@GALOISwidth}{\@tempdima}%
\else%
\setlength{\@GALOISwidth}{\@tempdimb}%
\fi%
\def\@GALOISrightarrow{\hbox to\@GALOISwidth
{\@GALOIScolor{#3}\@GALOISrightarrowfill{#1}}}%
\def\@GALOISleftarrow{\hbox to\@GALOISwidth
{\@GALOIScolor{#5}\@GALOISleftarrowfill{#2}}}%
% compute half height of alpha/lower arrow
\settodepth{\@GALOISalphaarrowhalfheight}{$\mathord{#1}$}%
\settoheight{\@tempdima}{$\mathord{#1}$}%
\addtolength{\@GALOISalphaarrowhalfheight}{\@tempdima}%
\divide \@GALOISalphaarrowhalfheight by 2%
% compute half height of gamma/upper arrow
\settodepth{\@GALOISgammaarrowhalfheight}{$\mathord{#2}$}%
\settoheight{\@tempdima}{$\mathord{#2}$}%
\addtolength{\@GALOISgammaarrowhalfheight}{\@tempdima}%
\divide \@GALOISgammaarrowhalfheight by 2%
```

```
% compute the distance between the two arrows \@GALOISGap =
%   \max(\@GALOISalphaarrowhalfheight,
%        \@GALOISgammaarrowhalfheight)+\GaloisArrowsSep
\ifdim\@GALOISalphaarrowhalfheight>\@GALOISgammaarrowhalfheight%
\setlength{\@GALOISGap}{\@GALOISalphaarrowhalfheight}%
\else%
\addtolength{\@GALOISGap}{\@GALOISgammaarrowhalfheight}%
\fi%
\addtolength{\@GALOISGap}{\GaloisArrowsSep}%
% lift from the stems thickness
\addtolength{\@GALOISGap}{\GaloisArrowThickness }%
\addtolength{\@GALOISGap}{\GaloisArrowThickness }%
% compute height \@GALOISheight depth \@GALOISdepth
% and total height \@GALOIStotalheight  of the picture
\settodepth{\@GALOISalphadepth}{\@GALOISalphatag}%
\settoheight{\@GALOISalphaheight}{\@GALOISalphatag}%
\settodepth{\@GALOISgammadepth}{\@GALOISgammatag}%
% compute depth \@GALOISdepth of the picture
% \@GALOISdepth = \@GALOISalphadepth
%               + \@GALOISalphaheight % vertical size of alpha tag
%               + \GaloisArrowTagSep  % between top of tag and arrow
\setlength{\@GALOISdepth}{\@GALOISalphadepth}%
\addtolength{\@GALOISdepth}{\@GALOISalphaheight}%
\addtolength{\@GALOISdepth}{\GaloisArrowTagSep}%
%  lift from the stem thickness
\addtolength{\@GALOISdepth}{-\GaloisArrowThickness }%
% compute height \@GALOISheight of the picture
\setlength{\@GALOISheight}{\@GALOISGap}%
\addtolength{\@GALOISheight}{\GaloisArrowTagSep}%
\addtolength{\@GALOISheight}{\@GALOISgammadepth}%
\settoheight{\@tempdima}{\@GALOISgammatag}%
\addtolength{\@GALOISheight}{\@tempdima}%
% compute  total height \@GALOIStotalheight of the picture
% \@GALOIStotalheight = \@GALOISdepth + \@GALOISheight
\setlength{\@GALOIStotalheight}{\@GALOISdepth}%
\addtolength{\@GALOIStotalheight}{\@GALOISheight}%
% put alpha arrow
\Galois@put(0pt,0pt-\@GALOISalphaarrowhalfheight){\@GALOISrightarrow}%
% put gamma arrow
\Galois@put(0pt,\@GALOISGap-\@GALOISalphaarrowhalfheight){\@GALOISleftarrow}%
% put alpha
\setlength{\@tempdima}{\@GALOISwidth}%
\settowidth{\@tempdimb}{\@GALOISalphatag}%
\addtolength{\@tempdima}{-\@tempdimb}%
\divide\@tempdima by 2%
\Galois@put(\@tempdima,\@GALOISalphadepth-\@GALOISdepth){\@GALOISalphatag}%
% put gamma
\setlength{\@tempdima}{\@GALOISwidth}%
\settowidth{\@tempdimb}{\@GALOISgammatag}%
\addtolength{\@tempdima}{-\@tempdimb}%
\divide\@tempdima by 2%
\setlength{\@tempdimb}{\@GALOISalphadepth}%
\addtolength{\@tempdimb}{\@GALOISalphaheight}%
\addtolength{\@tempdimb}{\GaloisArrowTagSep}%
```

```
\addtolength{\@tempdimb}{\GaloisArrowTagSep}%
\addtolength{\@tempdimb}{\@GALOISGap}%
\addtolength{\@tempdimb}{\@GALOISgammadepth}%
\Galois@put(\@tempdima,\@tempdimb-\@GALOISdepth){\@GALOISgammatag}%
\rule[-\@GALOISdepth]{0pt}{\@GALOIStotalheight}% set depth and height
\hspace*{\@GALOISwidth}% set width
}}}%
```

\galois `\galois{a}{g}` is $\xleftarrow[a]{g}$.

```
\newcommand{\galois}{\@GALOIS{\rightarrow}{\leftarrow}}%
```

\galoiS `\galoiS{a}{g}` is $\xleftarrow[a]{g}\!\!\!\!\rightarrow$ (a onto, g one-to-one, $a \circ g = 1$):

```
\def\@GALOISmytwoheadrightarrow{\rlap{$\:\,{\rightarrow}$}{\longrightarrow}}%
\def\@GALOISmytwoheadrightarrow{\protect\@GALOISmytwoheadrightarrow}%
\newcommand{\galoiS}{\@GALOIS{\@GALOISmytwoheadrightarrow}{\leftarrow}}%
```

\galoiSr `\galoiSr{a}{g}` is $\xleftarrow[a]{g}\!\!\!\!\twoheadrightarrow$.

```
\def\@GALOISmytwoheadrightarrowreduc{\rlap{\smash{\hskip1ex\raisebox{0.815ex}%
{\tiny$\equiv$}}}\rlap{$\:\,{\rightarrow}$}{\longrightarrow}}%
\def\@GALOISmytwoheadrightarrowreduc{\protect\@GALOISmytwoheadrightarrowreduc}%
\newcommand{\galoiSr}{\@GALOIS{\@GALOISmytwoheadrightarrowreduc}{\leftarrow}}%
```

\Galois `\Galois{a}{g}` is $\xleftarrow[a]{g}$ (a one-to-one, g onto, $g \circ a = 1$):

```
\def\@GALOISmytwoheadleftarrow{\rlap{$\:{\leftarrow}$}{\longleftarrow}}%
\def\@GALOISmytwoheadleftarrow{\protect\@GALOISmytwoheadleftarrow}%
\newcommand{\Galois}{\@GALOIS{\rightarrow}{\@GALOISmytwoheadleftarrow}}%
```

\GaloiS `\GaloiS{a}{g}` is $\xleftarrow[a]{g}\!\!\!\!\twoheadrightarrow$ (a bijective with inverse g).

```
\newcommand{\GaloiS}{\@GALOIS%
{\@GALOIStwoheadrightarrow}{\@GALOISmytwoheadleftarrow}}%
```

```
⟨/package⟩
```

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