

Parameterization of copula functions for bivariate survival data in the **surrosurv** package (v. 1.1.26). Modelling and simulation

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Let define the joint survival function of S and T via a copula function:

$$S(s, t) = P(S > s, T > t) = C(u, v)|_{u=S_S(s), v=S_T(t)}, \quad (1)$$

where $S_S(\cdot) = P(S > s)$ and $S_T(\cdot) = P(T > t)$ are the marginal survival functions of S and T .

Modelling

In the case of possibly right-censored data, the individual contribution to the likelihood is

- $S(s, t) = C(u, v)|_{S_S(s), S_T(t)}$ if S is censored at time s and T is censored at time t ,
- $-\frac{\partial}{\partial t} S(s, t) = \frac{\partial}{\partial v} C(u, v)|_{S_S(s), S_T(t)} f_T(t)$ if S is censored at time s and $T = t$,
- $-\frac{\partial}{\partial s} S(s, t) = \frac{\partial}{\partial u} C(u, v)|_{S_S(s), S_T(t)} f_S(s)$ if $S = s$ and T is censored at time t ,
- $\frac{\partial^2}{\partial s \partial t} S(s, t) = \frac{\partial^2}{\partial u \partial v} C(u, v)|_{S_S(s), S_T(t)} f_S(s) f_T(t)$ if $S = s$ and $T = t$.

Clayton copula

The bivariate Clayton [1978] copula is defined as

$$C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad \theta > 0. \quad (2)$$

The first derivative with respect to u is

$$\begin{aligned} \frac{\partial}{\partial u} C(u, v) &= (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1+\theta}{\theta}} u^{-(1+\theta)} \\ &= \left[\frac{C(u, v)}{u} \right]^{1+\theta}. \end{aligned} \quad (3)$$

The second derivative with respect to u and v is

$$\frac{\partial^2}{\partial u \partial v} C(u, v) = (1 + \theta) \frac{C(u, v)^{1+2\theta}}{(uv)^{1+\theta}}. \quad (4)$$

The Kendall [1938]'s tau for the Clayton copula is

$$\tau = \frac{\theta}{\theta + 2}. \quad (5)$$

Plackett copula

The bivariate Plackett [1965] copula is defined as

$$C(u, v) = \frac{[Q - R^{1/2}]}{2(\theta - 1)}, \quad \theta > 0, \quad (6)$$

with

$$\begin{aligned} Q &= 1 + (\theta - 1)(u + v), \\ R &= Q^2 - 4\theta(\theta - 1)uv. \end{aligned} \quad (7)$$

Given that

$$\frac{\partial}{\partial u} Q = \theta - 1, \quad (8)$$

$$\begin{aligned} \frac{\partial}{\partial u} R &= 2(\theta - 1) \left(1 - (\theta + 1)v + (\theta - 1)u \right) \\ &= 2(\theta - 1)(Q - 2\theta v), \end{aligned} \quad (9)$$

the first derivative of $C(u, v)$ with respect to u is

$$\begin{aligned} \frac{\partial}{\partial u} C(u, v) &= \frac{1}{2} \left[1 - \frac{1 - (\theta + 1)v + (\theta - 1)u}{R^{1/2}} \right] \\ &= \frac{1}{2} \left[1 - \frac{Q - 2\theta v}{R^{1/2}} \right]. \end{aligned} \quad (10)$$

By defining

$$f = Q - 2\theta v, \quad (11)$$

$$g = R^{1/2} \quad (12)$$

and given that

$$f' = \frac{\partial}{\partial v} f = -(\theta + 1), \quad (13)$$

$$\begin{aligned} g' &= \frac{\partial}{\partial v} g = \frac{\theta - 1}{R^{1/2}} \left(1 - (\theta + 1)u + (\theta - 1)v \right) \\ &= \frac{\theta - 1}{R^{1/2}} (Q - 2\theta u), \end{aligned} \quad (14)$$

then, the second derivative with respect to u and v is (see Appendix A for full details)

$$\begin{aligned} \frac{\partial^2}{\partial u \partial v} C(u, v) &= -\frac{f'g - fg'}{2g^2} \\ &= \frac{\theta}{R^{3/2}} \left[1 + (\theta - 1)(u + v - 2uv) \right] \\ &= \frac{\theta}{R^{3/2}} \left[Q - 2(\theta - 1)uv \right]. \end{aligned} \quad (15)$$

The Kendall's tau for the Plackett copula cannot be computed analytically and is obtained numerically.

Gumbel–Hougaard copula

The bivariate Gumbel [1960]–Hougaard [1986] copula is defined as

$$C(u, v) = \exp \left(-Q^\theta \right), \quad \theta \in (0, 1), \quad (16)$$

with

$$Q = (-\ln u)^{1/\theta} + (-\ln v)^{1/\theta}. \quad (17)$$

Given that

$$\frac{\partial}{\partial u} Q = -\frac{(-\ln u)^{1/\theta-1}}{\theta u}, \quad (18)$$

then, the first derivative with respect to u is

$$\frac{\partial}{\partial u} C(u, v) = \frac{(-\ln u)^{1/\theta-1}}{u} C(u, v) Q^{\theta-1} \quad (19)$$

and the second derivative with respect to u and v is

$$\frac{\partial^2}{\partial u \partial v} C(u, v) = \frac{\left[(-\ln u)(-\ln v)\right]^{1/\theta-1}}{uv} C(u, v) Q^{\theta-2} \left[\frac{1}{\theta} - 1 + Q^\theta\right]. \quad (20)$$

The Kendall's tau for the Gumbel–Hougaard copula is

$$\tau = 1 - \theta. \quad (21)$$

Simulation

Clayton copula

The function `simData.cc()` generates data from a Clayton copula model. First, the time value for the surrogate endpoint S is generated from its (exponential) marginal survival function:

$$S = -\log(U_S/\lambda_S), \quad \text{with } U_S := S_S(S) \sim U(0, 1). \quad (22)$$

Then, the time value for the true endpoint T is generated conditionally on the value s of S . The conditional survival function of $T \mid S$ is

$$S_{T|S}(t \mid s) = \frac{-\frac{\partial}{\partial s} S(s, t)}{-\frac{\partial}{\partial s} S(s, 0)} = \frac{\frac{\partial}{\partial u} C(u, v)}{\frac{\partial}{\partial u} C(u, 1)} \quad (23)$$

As the Clayton copula is used, we get (see Equation 3)

$$\begin{aligned} S_{T|S}(t \mid s) &= \left[\frac{C(S_S(s), S_T(t))}{C(S_S(s), 1)} \right]^{1+\theta} = \left[\frac{U_S^{-\theta} + S_T(t)^{-\theta} - 1}{U_S^{-\theta}} \right]^{-\frac{1+\theta}{\theta}} \\ &= \left[1 + U_S^\theta (S_T(t)^{-\theta} - 1) \right]^{-\frac{1+\theta}{\theta}} \end{aligned} \quad (24)$$

By generating uniform random values for $U_T := S_{T|S}(T \mid s) \sim U(0, 1)$, the values for $T \mid S$ are obtained as follows:

$$\begin{aligned} U_T &= \left[1 + U_S^\theta (S_T(T)^{-\theta} - 1) \right]^{-\frac{1+\theta}{\theta}} \\ S_T(T) &= \left[\left(U_T^{-\frac{\theta}{1+\theta}} - 1 \right) U_S^{-\theta} + 1 \right]^{-1/\theta} \\ T &= -\log(S_T(T)/\lambda_T). \end{aligned} \quad (25)$$

Gumbel–Hougaard copula

The function `simData.gh()` generates data from a Gumbel-Hougaard copula model. First, the time value for the surrogate endpoint S is generated from its (exponential) marginal survival function:

$$S = -\log(U_S/\lambda_S), \quad \text{with } U_S := S_S(S) \sim U(0, 1). \quad (26)$$

The conditional survival function of $T \mid S$ is (see Equation 19)

$$\begin{aligned} S_{T|S}(t \mid s) &= \exp \left(Q(S_S(s), 1)^\theta - Q(S_S(s), S_T(t))^\theta \right) \left[\frac{Q(S_S(s), S_T(t))}{Q(S_S(s), 1)} \right]^{\theta-1} \\ &= \exp \left(-\log U_S - \left[(-\log U_S)^{\frac{1}{\theta}} + (-\log S_T(t))^{\frac{1}{\theta}} \right]^\theta \right) \left[1 + \left(\frac{\log S_T(t)}{\log U_S} \right)^{\frac{1}{\theta}} \right]^{\theta-1} \end{aligned} \quad (27)$$

By generating uniform random values for $U_T := S_{T|S}(T \mid s) \sim U(0, 1)$, the values of $S_T(T)$ are obtained by numerically solving

$$U_T - \exp \left(-\log U_S - \left[(-\log U_S)^{\frac{1}{\theta}} + (-\log S_T(T))^{\frac{1}{\theta}} \right]^\theta \right) \left[1 + \left(\frac{\log S_T(T)}{\log U_S} \right)^{\frac{1}{\theta}} \right]^{\theta-1} = 0 \quad (28)$$

and then the times $T \mid S$ are

$$T = -\log(S_T(T)/\lambda_T). \quad (29)$$

References

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A Second Derivative of the Plackett Copula

Let $f = Q - 2\theta v$, and $g = R^{1/2}$, with $Q = 1 + (\theta - 1)(u + v)$ and $R = Q^2 - 4\theta(\theta - 1)uv$. Hence,

$$f' = \frac{\partial}{\partial v} f = -(\theta + 1), \quad (30)$$

$$g' = \frac{\partial}{\partial v} g = \frac{\theta - 1}{R^{1/2}} (Q - 2\theta u). \quad (31)$$

Then, the second derivative of $C(u, v)$ with respect to u and v is

$$\begin{aligned} \frac{\partial^2}{\partial u \partial v} C(u, v) &= -\frac{f'g - fg'}{2g^2} = \frac{fg' - f'g}{2g^2} \\ &= \frac{1}{R} \left[\frac{\theta - 1}{2R^{1/2}} (Q - 2\theta u) (Q - 2\theta v) + (\theta + 1)R^{1/2} \right] \\ &= \frac{1}{2R^{3/2}} \left[(\theta - 1) (Q - 2\theta u) (Q - 2\theta v) + (\theta + 1)R \right] \\ &= \frac{1}{2R^{3/2}} \left[(\theta - 1) (Q^2 + 4\theta^2 uv - 2\theta Q(u + v)) + (\theta + 1) (Q^2 - 4\theta(\theta - 1)uv) \right] \\ &= \frac{1}{2R^{3/2}} \left[((\theta - 1)Q^2 - 4\theta^2(\theta - 1)uv - 2\theta Q(\theta - 1)(u + v)) + ((\theta + 1)Q^2 - 4\theta(\theta^2 - 1)uv) \right] \end{aligned} \quad (32)$$

Since $(u + v)(\theta - 1) = Q - 1$, then

$$\begin{aligned} \frac{\partial^2}{\partial u \partial v} C(u, v) &= \frac{1}{2R^{3/2}} \left[2\theta Q^2 - 4\theta(\theta - 1)uv - 2\theta Q(Q - 1) \right] \\ &= \frac{1}{2R^{3/2}} \left[2\theta Q - 4\theta(\theta - 1)uv \right] \\ &= \frac{\theta}{R^{3/2}} \left[Q - 2(\theta - 1)uv \right]. \end{aligned} \quad (33)$$